Analysis Of PDE
Lecture 2

Recap: PDE linear if of the form $\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x)$

Say a linear PDE is homogeneous if $f \equiv 0$.

Theorem (Picard-Lindelöff): (Thm 2.1)
Fix $U \subset \mathbb{R}^n$ open. $f: U \to \mathbb{R}^n$ given. Consider,
$$\boxed{\dot u(t) = f(u(t)),\ u(0) = u_0 \in U} \quad (1)$$

Suppose $\exists\, r, K > 0$ s.t. $B_r(u_0) \subset U$ and $|f(x) - f(y)| \le K|x-y|$ $\forall x,y \in B_r(u_0)$.
Then $\exists\, \varepsilon = \varepsilon(r,K)$ and $\exists!$ C^1 function $u: (-\varepsilon,\varepsilon) \to U$ solving (1).

Proof (sketch): If $u \in C^1$ solves (1), then by FTC $\Rightarrow u(t) = u_0 + \int_0^t f(u(s))\,ds$ — (2)
Conversely, if $u \in C^0$ solution to (2), then by the FTC it solves (1) $\Rightarrow$ reduction in regularity & apply fixed point methods. Thus, u, if it exists, is a fixed point of the map:
$$G(u(t)) = u_0 + \int_0^t f(u(s))\,ds.$$

Let $S = \{u : (-\varepsilon,\varepsilon) \to \overline{B_{r/2}(u_0)} : u \in C^0\}$

NTP: • S is a complete metric space.
• $G: S \to S$ is a contraction for sufficiently small ε.
$\Longrightarrow$ conclude by the CMT (Sheet 1).

Remarks: (1) Can't be global.
Ex: $\dot u(t) = u(t)^2$, $u(0) = u_0 > 0$

(2) Possible apply to $\dot u(t) = \sqrt{|u(t)|}$, $u(0)=0$. (non-unique case) find two sol^{ns}, note can apply Peano theorem to deduce existence.

Assume that $f \in C^\infty(U)$. We have $\dot u = f(u(t))$ and have $u \in C^1((-\varepsilon,\varepsilon))$. Chain rule,
$\ddot u(t) = Df(u(t))\cdot \dot u(t) = f_2(u(t), \dot u(t))$
$\Rightarrow u \in C^2$. Similarly, $\dddot u(t) = \frac{df_2}{dt} \in C^0$

$\Rightarrow u \in C^3$. Can continue like so to deduce that $u \in C^\infty$ (given $f \in C^\infty$).
In principle, given $u_0 = u(0)$ we can determine $u^{(k)}(0) = \underbrace{F_k(u_0, \dot u_0, \ldots, u^{(k-1)})}_{\text{polynomial}}\big|_{t=0}$

so we can write $\sum_{k\ge 0} \frac{u^{(k)}(0)}{k!} t^k$

Call this a "formal power series solution."

[Q]: Does $u(t) \equiv \sum_{k\ge0} \frac{u^{(k)}(0)}{k!} t^k$ in a nbhd of 0? for simple ODEs.

Thm 2.2 (Cauchy – Kovalevskaya)
1842 – 1875

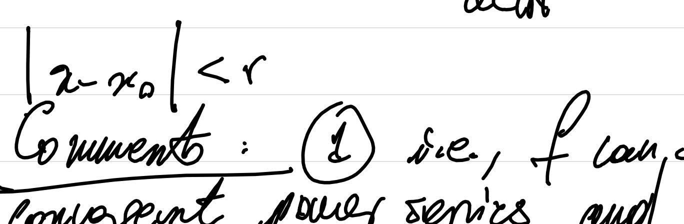
If $f(u)$ is real analytic in a nbhd of u_0, then the series $\sum_{k\ge0} \frac{u^{(k)}(0)}{k!}$ converges in a nbhd of $t=0$ to the unique sol^{n} of (1) given by Picard-Lindeloff.

2.2: Real analyticity (RA) and majorants.

Suppose $f: (-\varepsilon,\varepsilon) \to \mathbb{R}$ is smooth $\Rightarrow f^{(n)}(0)$ exists $\forall n \ge 0$.

[Q]: Does $\sum_{n\ge0} \frac{f^{(n)}(0)}{n!} x^n$ converge to $f(x)$ for $|x| \le \delta$?

[A] No: Ex $f(x) = \begin{cases} e^{-1/x}, & x>0 \\ 0, & x \le 0\end{cases}$



Can show that $f^{(n)}(0) = 0\ \forall n \ge 0$.

Def: Let $U \subset \mathbb{R}^n$ open, and $f: U \to \mathbb{R}$. Say f is real analytic if $\exists\, r>0$ and $f_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{N}^n$ s.t. $f(x) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha (x-x_0)^\alpha$ for all $|x-x_0| < r$

Comments: (1) i.e., f can be written as a convergent power series and $f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!}$
(2) if f is real analytic at a point $x_0 \in U \Rightarrow f$ is real analytic in a nbhd of $x_0 \in U$.
(3) Denote the set of RA functions on U by $C^\omega(U)$.
(4) If $f \in C^\omega(U)$, then $f \in C^\infty(U)$.
Exercise: justify term by term diff^n; (Hint: use Weierstrass M-test).
(5) if $f \in C^\omega(U)$ and U a connected open set in $\mathbb{R}^n$, then f is uniquely determined in U if we know $D^\alpha f(x)$ $\forall \alpha \in \mathbb{N}^n$ and some $x \in U$.

Ex: Show $f(x) = 1/x$, $f(x) = x^{1/2}$ are RA for $x > 0$.

Example: Recall $\frac{1}{1-x} = \sum_{k\ge0} x^k$, $|x|<1$ in 1dim.
Let $r>0$ be positive. Consider:
$$f(x) = \frac{r}{r - (x_1+\cdots+x_n)} = \frac{1}{1 - \frac{(x_1+\cdots+x_n)}{r}}$$
$$= \sum_{n\ge0} \frac{(x_1+\cdots+x_n)^n}{r^n} \text{ when}$$
$$|x_1+\cdots+x_n| \le \left(\sum |x_i|^2\right)^{1/2}\cdot\sqrt{n} \le |x|\cdot\sqrt{n} < r$$
By multinomial theorem (Sheet 1):
$$f(x) = \sum_{k\ge0} \frac{1}{r^k}\Big(\sum_{|\alpha|=k} \underbrace{\binom{|\alpha|}{\alpha}}_{\text{"}\frac{|\alpha|!}{\alpha_1!\cdots\alpha_n!}\text{"}} x^\alpha\Big)$$
$$= \sum_\alpha \frac{|\alpha|!}{\alpha!}\frac{x^\alpha}{r^{|\alpha|}}$$
So $f(x) = \sum_\alpha f_\alpha x^\alpha$ where $f_\alpha = \frac{|\alpha|!}{\alpha!}\frac{1}{r^{|\alpha|}}$

and $D^\alpha f(0) = \frac{|\alpha|!}{r^{|\alpha|}}$. This series is absolutely convergent near 0.
$$\sum \frac{|\alpha|!}{\alpha!}\frac{|x^\alpha|}{r^{|\alpha|}} = \sum_{k\ge0}\left(\frac{|x_1|+\cdots+|x_n|}{r}\right)^k < \infty.$$
since $|x_1|+\cdots+|x_n| \le \|x\|\cdot\sqrt{n} < r$.

Def^n: Let $f = \sum_\alpha f_\alpha x^\alpha$, $g = \sum_\alpha g_\alpha x^\alpha$ be two power series. We say g majorises f or g is a majorant of f written $g \gg f$ if $g_\alpha \ge |f_\alpha|$ $\forall \alpha$. (If vector-valued $g^i \gg f^i$ $\forall i$)

Lemma 2.3: (Properties of Majorants)
(i) If $g \gg f$ and g converges for $\|x\| < r$, then f converges for $\|x\| < r$.
(ii) If $f = \sum f_\alpha x^\alpha$ converges for $\|x\| < r$, then for $s \in (0, r/\sqrt{n})$ $\exists$ a majorant of f which converges (for $\|x\| < s/\sqrt{n}$).

Proof: (i) $\sum_{|\alpha|\le k} |f_\alpha x^\alpha| = \sum_{|\alpha|\le k} |f_\alpha|\cdot|x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n}$
$$\le \sum_{|\alpha|\le k} g_\alpha\cdot|x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n} \le \sum_\alpha g_\alpha |x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n}$$
$= g(\tilde x)$, where $\tilde x = (|x^1|, \ldots, |x^n|)$.
So $\|\tilde x\| = \|x\|$ and so if $\|x\| < r \Rightarrow \|\tilde x\| = \|x\| < r$
$\Rightarrow g(\tilde x)$ converges at $\tilde x$.
$\Rightarrow$ uniform bound on partial sums, take $k \to \infty$, done. ✓

(ii) Let $s \in (0, r/\sqrt{n})$ and set $y = (s,\ldots,s) = s(1,\ldots,1)$, then $\|y\| = s\sqrt{n}$.
By assumption, $f(y) = \sum_\alpha f_\alpha y^\alpha$ converges as $\|y\| = s\cdot\sqrt{n} < r$. So $\exists\, C > 0$ s.t.
$|f_\alpha y^\alpha| \le C$ $\forall \alpha \Rightarrow |f_\alpha| \le \frac{C}{|y^\alpha|} = \frac{C}{|y_1|^{\alpha_1}\cdots|y_n|^{\alpha_n}}$
$= \frac{C}{|s|^{|\alpha|}} \le \frac{C}{|s|^{|\alpha|}}\frac{|\alpha|!}{\alpha!}$. So define $g(x) = \frac{Cs}{s-(x_1+\cdots+x_n)}$
$= C\cdot\sum_\alpha \frac{|\alpha|!}{\alpha!\, s^{|\alpha|}} x^\alpha$. This series now converges for $\|x\| < s/\sqrt{n}$ (and clearly majorises f).

Analysis of PDE
Lecture 3

Theorem: Suppose $U \subset \mathbb{R}^n$ is open $u_0 \in U$. If $f : U \to \mathbb{R}$ is real analytic near u_0 and $u(t)$ is the unique sol^{n} of $\begin{cases} \dot{u}(t) = f(u(t)) \\ u(0) = u_0 \end{cases}$ given by Picard-Lindeloff, then u is also real analytic near $t=0$.

Comments: (1) A function is Real Analytic on an open set U if it is RA at all points $x_0 \in U$
(2) f is RA on an open set $U \Leftrightarrow$ for any compact set $K \subset U$ $\exists C = C(K), r > 0$, s.t.
$$\sup_{x \in K} |D^\alpha f(x)| \leq C(K) \cdot \frac{|\alpha|!}{r^{|\alpha|}}$$
(3) RA is a local property.

Proof: (Method of Majorants): WLOG $u_0 = 0$, simplicity $n=1$. We need to find the series coefficients. So $\dot{u} = f(u) \Rightarrow \dot{u}(0) = f(0)$
$\Rightarrow u_1 = f(0)$. Next, $\ddot{u}(t) = f'(u) \cdot \dot{u}(t)$
$\Rightarrow \ddot{u}(0) = f'(0) \cdot f(0) \Rightarrow u_2 = \frac{1}{2} f'(0) \cdot f(0)$.
Similarly $\dddot{u}(t) = f''(u(t)) \cdot f(u(t)) \dot{u}(t)$
$+ (f'(u(t)))^2 \dot{u}(t)$
$\Longrightarrow u_3 = \frac{1}{3!}$ (").

Iterating, $u_k = P_k(f(0), f'(0), f''(0), \dots, f^{(k-1)}(0))$, a polynomial of k variables with <u>non-negative</u> coefficients.
E.g.: $P_1(x) = x$, $P_2(x,y) = \frac{1}{2}(x \cdot y)$,
$P_3(x,y,z) = \frac{1}{3}(x^2 z + xy^2)$.
Since f is RA we have $f(v) = \sum_{k \geq 0} f_k v^k$
with $f_k = \frac{f^{(k)}(0)}{k!} \Rightarrow f^{(k)}(0) = k! f_k$.

So, $u_k = Q_k(f_0, f_1, \dots, f_{k-1})$, a polynomial with non-negative coefficients. This polynomial is "universal". Aim: Show that $\sum_{k \geq 0} u_k t^k$ in a nbhd of $t=0$ converges and solves the ODE.
Since f is analytic, we know $f(u) = \sum_{k \geq 0} f_k u^k$ converges for some small $|u| < r$, $r > 0$.
Fixing $s \in (0, r)$ we know from Lemma 2.3 that $\exists$ majorant of given by $g(u) = \sum g_k u^k$ s.t. $g(u) = \sum_{k \geq 0} \frac{C}{s^k} u^k = \frac{Cs}{s-u}$, $|u| < s$.
(C fixed). Consider the aux. ODE
$$\begin{cases} \dot{w}(t) = g(w(t)), \quad (*) \\ w(0) = 0. \end{cases}$$
$\frac{dw}{dt} = \frac{C \cdot s}{s - w(t)} \Rightarrow w(t) = s \pm \sqrt{s^2 - 2Cst}$, take negative value to agree with initial data.
Then, $w(t) = s - \sqrt{s^2 - 2Cst}$ solves $(*)$. This is RA for $|t| < \frac{s}{2C} \Rightarrow w(t) = \sum_{k \geq 0} w_k t^k$ converges for $|t| < \frac{s}{2C}$
$\Rightarrow w_k = Q_k(g_0, g_1, \dots, g_{k-1})$, Q_k universal polynomials. Claim now w majorises u. By construction, $g_k \geq |f_k|$ for all $k \geq 0$.
$$\begin{aligned} \Rightarrow w_k &= Q_k(g_0, \dots, g_{k-1}) \\ &\geq Q_k(|f_0|, \dots, |f_{k-1}|) \\ &\geq |Q_k(f_0, \dots, f_{k-1})| = |u_k|. \end{aligned}$$

By Lemma (1) last time, we know $\sum_{k \geq 0} u_k t^k$ converges for $|t| < \frac{s}{2C}$.
To conclude, $u(t) := \sum_{k \geq 0} u_k t^k$ and we need to check that $u(t)$ solves the ODE. Both sides of ODE are analytic so it suffices to check derivatives on each side agree to all orders at $t=0$ (done by construction) $\square$
Remarks: (1) Can extend to systems
$u_k = u_k^j = Q_k^j(D_u^\alpha f(0))_{|\alpha| \leq k-1}$
$w \longrightarrow w^j = w^1$ as before $\forall j$.
(2) In the non-autonomous case,
$\dot{u}(t) = f(u, t)$, $u(0) = 0$. Consider $v(t) = (u(t), t)$
then $\dot{v}(t) = (\dot{u}, 1) = (f(u,t), 1) = (f(v), 1)$
$= F(v)$
with $v(0) = 0 \Rightarrow$ Apply system C-K.

2.4 (CK) for PDEs.
Unknown $\underline{u} : \mathbb{R}^n \to \mathbb{R}^m$, some $r > 0$.
Consider $\partial_t \underline{u} = \sum_{j=1}^{n-1} \underline{B}_j(\underline{u}, x') \partial_{x_j} \underline{u} + \underline{C}(\underline{u}, x')$

$\underline{u}(x', t=0) = 0$ on $x' \in B_r^{n-1}(0)$ with $x' \in \mathbb{R}^{n-1}$ ($t = x^n$). We seek a sol^{n} to (3) on the subset $B_r^n(0) = \{x \in \mathbb{R}^n \mid \|x\| = \sqrt{\sum_{i=1}^n |x_i|^2} < r\}$.

Theorem (2.3): [CK for first order systems].
Suppose $\{B_1, \dots, B_{n-1}, C\}$ are RA. Then for some small $r > 0$, $\exists!$ real analytic f^{n} $\underline{u} = \sum_\alpha \underline{u}_\alpha x^\alpha$ that solves (3).

Idea: Compute $\underline{u}_\alpha = \frac{D_x^\alpha \underline{u}(0)}{\alpha!}$ in terms of $\{\underline{B}_j, \underline{C}\}$ and show that power series converges for small r. We use the PDE to find all derivatives.
Example: Consider $\begin{cases} u_t = v_x - f \\ v_t = -u_x \end{cases}$ with $u = v = 0$ on $\{t = 0\}$
BC's $\Rightarrow u(0,0) = v(0,0) = 0$. Aim: determine u_α for all α. By diff^{n} the BCs
$(\partial_x)^n u(x,0) = 0 = (\partial_x)^n v(x,0)$ $\forall n \geq 0$, i.e. $\alpha = (n, 0)$. Then from the PDE $\begin{cases} u_t(x,0) = 0 - f(x,0) \\ v_t(x,0) = 0 \end{cases}$
$$\Rightarrow \begin{aligned} (\partial_x)^n \partial_t u(x,0) &= -(\partial_x)^n (f(x,0)) \\ (\partial_x)^n \partial_t v(x,0) &= 0 \quad \forall n \in \mathbb{N} \text{ i.e. } \alpha = (n,1). \end{aligned}$$
Next, if $\alpha = (n, 2)$, use the PDE to get
$u_{tt}(x,0) = -f_t(x,0)$ and
$v_{tt}(x,0) = -v_{xx} + f_x = f_x$.
$$\Rightarrow \begin{aligned} (\partial_x)^n \partial_t^2 u(x,0) &= -(\partial_x)^n \partial_t f(x,0). \\ (\partial_x)^n \partial_t^2 v(x,0) &= (\partial_x)^{n+1} f(x,0). \end{aligned}$$
Iterate on the number of derivatives in t.

2.5 | Reduction to first order systems.

Example: $u : \mathbb{R}^3 \to \mathbb{R}$, satisfying:

$$u_{tt} = u \cdot u_{xy} - u_{xx} + u_t$$
$$u|_{t=0} = u_0(x,y),$$
$$u_t|_{t=0} = u_1(x,y)$$

Suppose u_0, u_1 are RA near $0 \in \mathbb{R}^2$.

Note (consider): $f(t,x,y) = u_0 + t u_1$ is RA near 0 in $\mathbb{R}^3$ and $f|_{t=0} = u_0$, $\partial_t f|_{t=0} = u_1$.

$w(t,x,y) = u - f$, then

$$u_{tt} = w \cdot w_{xy} - w_{xx} + w_t + f \cdot w_{xy} + f_{xy} w + F$$
$$F = f \cdot f_{xy} - f_{xx} + f_t, \quad w|_{t=0} = \partial_t w|_{t=0} = 0.$$

Observe F is RA and does not depend on w and its derivatives.

Let $\underline{x} = (x,y,t) = (x_1, x_2, x_3)$ and set $\underline{v} = (w, w_x, w_y, w_t)$.

Then, $v^1_t = w_t = v^4$, $v^2_t = w_{xt} = v^4_{x_1}$,

$v^3_t = w_{yt} = v^4_{x_2}$, $v^4_t = v^1 \cdot v^2_{x_2} - v^2_{x_1} + v^4 + f v^2_{x_2} + f_{xy} v^1 + F$.

Define: $B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ v^1 + f & 0 & 0 & 0 \end{bmatrix}$

$\underline{c} = \begin{bmatrix} v^4 \\ 0 \\ 0 \\ v^4 + f_{xy} v^1 + F \end{bmatrix} \Rightarrow \partial_{x_3} \underline{v} = \sum_{j=1}^{2} B_j \underline{v}_{x_j} + \underline{c}$

$\underline{v} = 0$

Now, B_j, $\underline{c}$ are RA functions of $\underline{x}, \underline{v} \Rightarrow$ apply CK

More generally, consider the scalar quasilinear problem:

$$\sum_{|\alpha|=k} a_\alpha (D^{k-1}u, \dots, u, x) D^\alpha u + a_0 (D^{k-1}u, \dots, u, x) = 0.$$

where $u : B_r(0) \subset \mathbb{R}^n \to \mathbb{R}$, $u = \partial_n u = \dots = (\partial_{x_n} u)^{k-1} = 0$ for $\|x'\| \le r$, $\{x_n = 0\}$.

Introduce $\underline{v} = \left(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1 x_n}, \dots, \frac{\partial^{k-1} u}{\partial x_n^{k-1}}\right)$

$\hookrightarrow$ all derivatives of u, $D^\alpha u$, $|\alpha| \le k-1$.

$= (v^1, \dots, v^m) \in \mathbb{R}^m$.

Goal: get a 1st order system in $\underline{v}$.

Express $\frac{\partial \underline{v}}{\partial x_n}$ in terms of $\underline{v}, \frac{\partial \underline{v}}{\partial x_p}$, $p = 1, \dots, n-1$.

First consider the case $j \in \{1, \dots, m-1\}$. If $j = 1$, then $v^1 = u$, so $\frac{\partial v^1}{\partial x_n} = \frac{\partial u}{\partial x_n} = v^l$ for some $l \in \{1, \dots, m\}$.

If $2 \le j \le m-1$ then $v^j = D^\alpha u$, for some multi-index $|\alpha| \le k-1$ s.t. $\alpha_n < k-1$.

So $\frac{\partial v^j}{\partial x_n} = \frac{\partial D^\alpha u}{\partial x_n} = \frac{\partial^{|\alpha|+1} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$

$\rightarrow$ if $|\alpha| \le k-2$ then $|\alpha|+1 \le k-1$, then $\frac{\partial v^j}{\partial x_n} = v^l$ for $l \in \{1, \dots, m\}$.

$\rightarrow$ if $|\alpha| = k-1$ and $\alpha_n < k-1$. Then there is a $p \ne n$ s.t. $\alpha_p \ge 1$. So $\frac{\partial v^j}{\partial x^n} = \frac{\partial}{\partial x_n}\left(\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}\right)$

$= \frac{\partial}{\partial x_p}\left(\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_p^{\alpha_p - 1} \dots \partial x_n^{\alpha_n}}\right) = \frac{\partial}{\partial x_p}(v^l)$, $l \in \{1, \dots, m\}$.

To compute $\frac{\partial v^m}{\partial x^n} = \frac{\partial}{\partial x_n}\left(\frac{\partial^{k-1} u}{\partial x_n^{k-1}}\right)$ use the PDE.

Recall the coeffs as $(\underline{v}, \underline{x})$ for $\underline{v} \in \mathbb{R}^m$, $\underline{x} \in \mathbb{R}^n$.

We assume $a_\alpha : B_\rho(0) \to \mathbb{R}$ where $B_\rho(0) \subseteq \mathbb{R}^m \times \mathbb{R}^n$.

and suppose $a_c := a_{(0, \dots, 0, k)}(\underline{0}, \underline{0}) \ne 0$.

Since a_α are real analytic near $\underline{0} \Rightarrow a_\alpha$ are cont$^{s} \Rightarrow a_{(0, \dots, k)}(\underline{z}, \underline{u}) \ne 0$ $\forall \|z\|^2 + \|w\|^2 < \delta$, $\delta \le \rho$.

Then $\frac{\partial^k u}{\partial x_n^k} = -\frac{1}{a_{(0, \dots, 0, k)}}\left(\sum_{\substack{|\alpha|=k, \\ \alpha_n < k}} a_\alpha D^\alpha u + a_0\right)$

The RHS can be written in terms of $\frac{\partial v^l}{\partial x_p}, \underline{v}$, for $p < n$.

Conclusion: if $a_{(0, \dots, 0, k)} \ne 0$ we have turned the scalar quasilinear PDE into a first order system (on which we can apply C-K).

Def^n: If $a_{(0, \dots, 0, k)}(0, \dots, 0) \ne 0$, then we say the plane $\{x^n = 0\}$ is non-characteristic (else, we call it characteristic).

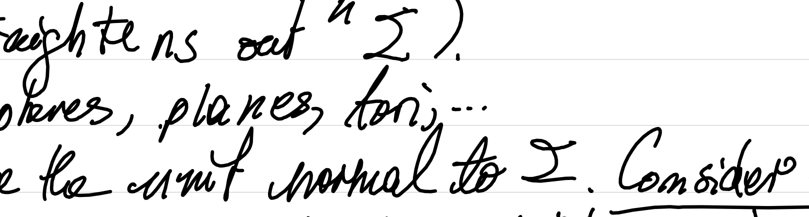
2.6) Exotic Boundary Conditions.

Def^n: $\Sigma \subset \mathbb{R}^n$ is a real analytic hypersurface near $x_0 \in \Sigma$ if $\exists\, \varepsilon > 0$ and a RA function $\Phi : B_\varepsilon(x_0) \to \mathcal{U} \subseteq \mathbb{R}^n$ open, $\underline{0} \in \mathcal{U}$ and defining $\underline{y} = \Phi(\underline{x})$ s.t. $\Phi(x_0) = \underline{0}$ and

(i): Φ is a bijection.

(ii): $\Phi^{-1} : \mathcal{U} \to B_\varepsilon(x_0)$ is real analytic

(iii): $\Phi(\Sigma \cap B_\varepsilon(x_0)) = \{y_n = 0\} \cap \mathcal{U}$



("Φ straightens out" Σ).

E.g's: spheres, planes, tori, ...

Let γ be the unit normal to Σ. Consider:

$$\sum_{|\alpha|=k} a_\alpha (D^{k-1}u, \dots, u, x) D^\alpha u + a_0 (D^{k-1}u, \dots, u, x) = 0.$$

$$u = (\gamma^i \partial_i u) = \dots = (\gamma^i \partial_i)^{k-1} u = 0 \text{ on } \Sigma.$$

Define $v(y) = u(\Phi^{-1}(y))$ for $y \in \mathcal{U}$.

$\Rightarrow u(x) = v(\Phi(x))$ for $x \in B_\varepsilon(x_0)$.

Chain-rule: $\Rightarrow \frac{\partial u}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial v}{\partial y^j} \frac{\partial \Phi^j}{\partial x^i}$ $(\Phi \in \mathbb{R}^n)$

So the PDE becomes $\sum b_\alpha D^\alpha v + b_0 = 0$ on $\mathcal{U}$.

where b_0, b_α depend on u and $D^\alpha u$ (for $|\alpha| \le k-1$) and also Φ and the BC's become

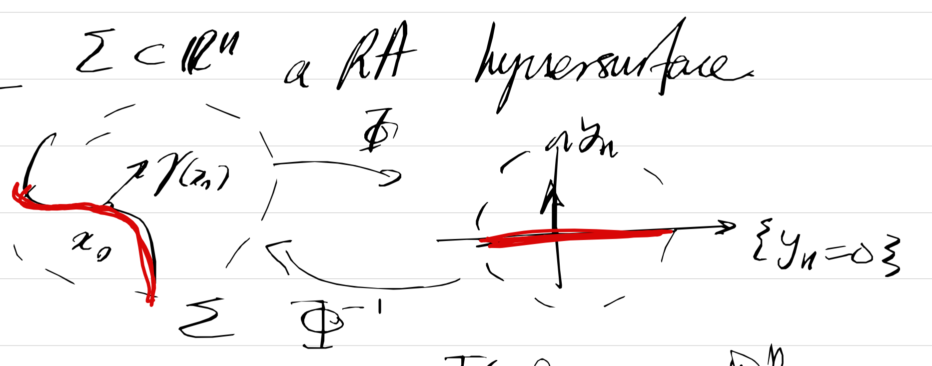
$$v = \partial_{y_n} v = \dots = (\partial_{y_n})^{k-1} v = 0 \text{ on } \{y_n = 0\}.$$

Since Φ is RA, so are b_α, b_0.

ANALYSIS OF PDE

LECTURE 5

Recap: $\Sigma \subset \mathbb{R}^n$ a RA hypersurface



$\Phi(\underline{x}) = \underline{y} \in \mathbb{R}^n$

What to solve:

$$(*)\quad \begin{cases} \sum_{|\alpha|=k} a_\alpha (D^{k-1}u, \dots, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, u, x) = 0 \\ u = \gamma^i \partial_i u = \dots = (\gamma^i \partial_i)^{k-1} u = 0 \text{ on } \Sigma \end{cases}$$

Define $v(y) = u(\Phi^{-1}(y)) \iff u(x) = v(\Phi(x))$, $x \in B_\varepsilon(x_0)$

Chain rule: $\dfrac{\partial u}{\partial x_i} = \sum_{j=1}^n \dfrac{\partial v}{\partial y_j} \dfrac{\partial \Phi^{(j)}}{\partial x_i}$

$\Rightarrow$ PDE $(*)$ $\sum_{|\alpha|=k} b_\alpha D^\alpha v + b_0 = 0$. and

$v = (\partial_{y_n} v) = \dots = (\partial_n)^{k-1} v = 0$ on $\{y_n = 0\}$

Check: $b_{(0,\dots,k)}(D^{k-1}v = 0, \dots, v = 0, y = 0) \overset{?}{=} 0$

i.e. determine if $\{y_n = 0\}$ is non-characteristic.

Note if $|\alpha| = k$ then $D^\alpha u = \dfrac{\partial^k}{\partial y_n^k} (D\Phi^{(n)})^\alpha$

$+\left(\text{terms not involving } \dfrac{\partial^k}{\partial y_n^k} v\right)$

Exercise: $k = 2$, $n = 2$, $\alpha = (0, 2)$.

$D^\alpha u = u_{x_2 x_2} = \partial_{x_2}(v_{y_1} \Phi^{(1)}_{x_2} + v_{y_2} \Phi^{(2)}_{x_2})$

$= \dots = v_{y_2 y_2} (\Phi^{(2)}_{x_2})(\Phi^{(2)}_{x_2}) + \text{l.o.t.}$ not involving $v_{y_2 y_2}$.

Thus, $b_{(0,\dots,k)} = \sum_{|\alpha|=k} a_\alpha \cdot (D\Phi^{(n)})^\alpha$ for the eqn $(*)$.

$\boxed{\text{Def}^{\underline{n}}\text{: Say } \Sigma \text{ is non-characteristic at } x_0 \in \Sigma \text{ if } b_{(0,\dots,k)} \sum_{|\alpha|=k} a_\alpha(0,\dots,k)(D\Phi^{(n)}(x_0))^\alpha \neq 0. \text{ Otherwise, it is characteristic.}}$

Remark: note $\Sigma = \{x \in \mathbb{R}^n \mid \Phi^{(n)}(x) = y_n = 0\}$

$\Rightarrow D\Phi^{(n)}(x) = c(x)\gamma(x)$, where γ is the unit normal of Σ.

$\Rightarrow D\Phi^{(n)}(x_0) = c(x_0)\gamma(x_0)$

$\Rightarrow$ non-characteristic condition is equivalent to

$\sum_{|\alpha|=k} a_{(0,\dots,k)}(0,\dots,x_0)\gamma(x_0)^\alpha \neq 0$.

Theorem (C-K on non-characteristic hypersurfaces)

Suppose $\Sigma \subset \mathbb{R}^n$ is a RA hypersurface. Consider $(*)$. Suppose a_0, a_α are RA near $x_0 \in \Sigma$ and that Σ is non-characteristic at x_0, then $\exists!$ RA sol$^{\underline{n}}$ to the problem in a nbhd of x_0.

Characteristic Surfaces: Consider the following linear operator $L = \sum_{i,j=1}^n a_{ij} \dfrac{\partial^2}{\partial x_i \partial x_j}$, $a_{ij} \in \mathbb{R}$.

Wlog $a_{ij} = a_{ji}$. Consider $\begin{cases} Lu = f \\ u = \gamma^i \partial_i u = 0 \text{ on } \Pi_\gamma = \{x \mid \underline{x} \cdot \underline{\gamma} = c\} \end{cases}$

i.e. BC's on the plane with normal vector γ and $\|\gamma\| = 1$.

We have Π_γ is non-characteristic iff $\sum_{i,j=1}^n a_{ij} \gamma^i \gamma^j \neq 0$, $\|\gamma\| = 1$.

Aim: find non-characteristic Π_γ.

Note LHS $= \langle A\gamma, \gamma \rangle$, $A = (a_{ij})$ symmetric

$\Rightarrow$ diagonalisable, i.e. $A = P^T \Lambda P$ where $P =$ unitary, $\Lambda =$ diagonal. So

$\langle A\gamma, \gamma \rangle = \langle P^T \Lambda P \gamma, \gamma \rangle = \langle \Lambda P\gamma, P\gamma \rangle$

$= \langle \Lambda v, v \rangle$ with $v = P\gamma$.

$\Rightarrow$ if $\{\lambda_i\}$ are evals of A, then the non-characteristic condition becomes

$\sum_{i=1}^n \lambda_i (v_i)^2 \neq 0$.

Case ①, all $\lambda_i > 0$ (or all $\lambda_i < 0$), since $v \neq 0$, then $\sum \lambda_i (v_i)^2 = 0$ is impossible

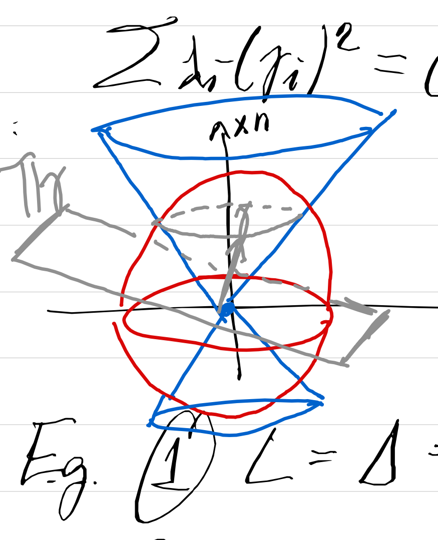
$\Rightarrow$ there are no characteristic hyperplanes Π_γ.

Call L an elliptic operator.

Case ②, one $\lambda_i < 0$ and the rest > 0, Call L a hyperbolic operator.

$\sum \lambda_i (\gamma_i)^2 = 0 \iff (\gamma_n)^2 = \sum_{i=1}^{n-1} (\gamma_i)^2$

and $\|\gamma\|^2 = 1$.



Cone $|x'|^2 - t^2 = |x'|^2 - x_n^2 = 0$.

Eg. ① $L = \Delta = \sum_{i=1}^n \partial_{x_i}^2 \to A = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ elliptic

② $L = -\partial_{x_n}^2 + \Delta \to$ hyperbolic since $A = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$

Aim: focus on different features of elliptic/hyperbolic operators

(Forget BC's, look for sol$^{\underline{n}}$s of the form $u(\underline{x}) = e^{i\underline{k}\cdot\underline{x}}$, $k \in \mathbb{R}^n$

i.e. wave-like solutions

$L(e^{i\underline{k}\cdot\underline{x}}) = -e^{i\underline{k}\cdot x} \sum_{j,l=1}^n a_{jl} k_j k_l \overset{?}{=} 0$ when

$L(e^{i\underline{k}\cdot\underline{x}}) = 0 ? \iff \sum a_{jl} k_j k_l$

If $\underline{k} = c \cdot \underline{\gamma}$, $\|\gamma\| = 1$, $\hookrightarrow (\iff) c^2 \sum a_{jl} \gamma_j \gamma_l = 0$.

If L is elliptic, then this is impossible, i.e., no wave-like solutions.

If L is hyperbolic, then we can have wave-like sol$^{\underline{n}}$s. i.e. $\sum a_{ij} \gamma_i \gamma_j = 0$, with $\|\gamma\| = 1$

$\Rightarrow u(x) = e^{i\lambda\gamma \cdot x}$ give ∞ family of sol$^{\underline{n}}$ indexed by $\lambda \in \mathbb{R}$.

As we take λ larger, $u'(x)$ can grow large.

$\Rightarrow$ sol$^{\underline{ns}}$ can be rough.

By contrast we will see that solutions to elliptic equations are smooth.

Analysis of PDE
Lecture 6

Consider
$$\left.\begin{aligned} u_{xx}+u_{yy}&=0\\ u(x,y=0)&=0\\ \partial_y u(x,y=0)&=0\end{aligned}\right\} \Rightarrow u(x,y)=0.$$

Exercise: given in typed lecture notes.
"Perturbed" data $u(x,y=0)=e^{-\sqrt{n}}\cos(nx)$,
$\partial_y(x,y=0)=0 \to 0$ as $n\to\infty$.

But RA solⁿ $|u_n(x,\varepsilon)|\to\infty$. This does not agree with part (iii) (continuous dependence) of Hadamard notion of Well Posedness.

3.1. Hölder spaces $C^{k,\gamma}$

Defⁿ Let $U\subset\mathbb{R}^n$ be open, $k\in\mathbb{N}$
$C^k(U)=\{u: U\to\mathbb{R} : u$ and $D^\alpha u$ are continuous $\forall|\alpha|\le k\}$.

Defineⁿ $C^k(\overline{U})=\{u\in C^k(U) \mid u, D^\alpha u$ are bounded and uniformly continuous on U $|\alpha|\le k\}$.

$$\|u\|_{C^k(\overline{U})}=\sum_{|\alpha|\le k}\sup|D^\alpha u|.$$

Idea: $u\in C^k(\overline{U})$ can be continuously extended to ∂U. Different to $\{u:\overline{U}\to\mathbb{R} \mid u$ and $D^\alpha u$ are continuous $|\alpha|\le k\}\supseteq C^k(\overline{U})$

Example sheet 2: $(C^k(\overline{U}), \|\cdot\|_{C^k(U)})$ is a Banach space.

Defⁿ: Say a function $u: U\to\mathbb{R}$ is Hölder continuous of index $\gamma\in(0,1]$ if $\exists C\ge 0$ s.t. $|u(x)-u(y)|\le C\cdot|x-y|^\gamma$ $\forall x,y\in U$.
If $\gamma=1$, then called Lipschitz continuous.
Ex: if $\gamma>1$, then u is constant.

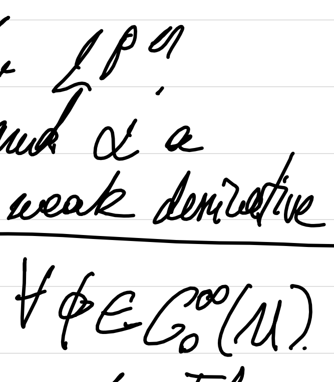
Defⁿ: For $\gamma\in(0,1]$ we say
$$C^{0,\gamma}(\overline{U})=\{u\in C^0(\overline{U}) \mid u \text{ is } \gamma\text{-Hölder continuous}\}$$
is the γ-Hölder space. Define the γ-Hölder seminorm by:
$$[u]_{C^{0,\gamma}(\overline{U})}:=\sup_{\substack{x,y\in U\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\gamma}\quad\left(\begin{array}{c}\text{smallest}\\ C \text{ in } (*)\end{array}\right).$$
Since constant functions vanish under $[\cdot]_{C^{0,\gamma}(\overline{U})}$ we add the $\|u\|_{C^{0,\gamma}(\overline{U})}:=[u]_{C^{0,\gamma}(\overline{U})}+\|u\|_{C^0(\overline{U})}$.

Exercise: $(C^{0,\gamma}(\overline{U}), \|\cdot\|_{C^{0,\gamma}(\overline{U})})$ is a Banach space. We extend to higher order derivatives. Defⁿ
$$C^{k,\gamma}(\overline{U}):=\{u\in C^k(\overline{U}) \mid D^\alpha u\in C^{0,\gamma}(\overline{U})\ \forall|\alpha|=k\}.$$
$$\|u\|_{C^{k,\gamma}(\overline{U})}:=\|u\|_{C^k(\overline{U})}+\sum_{|\alpha|=k}[D^\alpha u]_{C^{0,\gamma}(\overline{U})}.$$
Exercise: $(C^{k,\gamma}(\overline{U}), \|\cdot\|_{C^{k,\gamma}(\overline{U})})$ is a Banach space.

3.2 The Lebesgue spaces

Let $U\subset\mathbb{R}^n$ open and suppose $1\le p\le\infty$.
Defⁿ: $L^p(U)=\{u: U\to\mathbb{R}$ measurable $\mid \|u\|_{L^p(U)}<\infty\}/\sim$
where $\|u\|_{L^p(U)}=\left(\int_U|u(x)|^p dx\right)^{1/p}$ if $1\le p<\infty$ and
$$\|u\|_{L^\infty(U)}=\operatorname*{ess\,sup}_{x\in U}|u(x)|=\inf\{C\ge 0 \mid |u(x)|\le C \text{ pointwise a.e.}\}$$
if $p=\infty$ and where we quotient out by the equivalence relation $u_1\sim u_2$ if $u_1=u_2$ a.e.

Ex: $(L^p(U), \|\cdot\|_{L^p(U)})$ is a Banach space.
We also define local versions of L^p spaces. We say $u\in L^p_{loc}(U)$ if $u\in L^p(V)$ for every $V\subset\subset U$. Here "$V\subset\subset U$" reads "V compactly contained in U" which means $\exists$ a compact set K s.t. $V\subset K\subset U$. Thus
$$L^p_{loc}(U)=\bigcap_{V\subset\subset U}L^p(V)$$
Note, ① $L^p_{loc}(U)$ is not a Banach (is Fréchet).
② allows us to avoid the boundary.
e.g.: $u(x)\equiv 1\in L^1_{loc}(\mathbb{R})$
$\notin L^1(\mathbb{R})$.
e.g.: $u(x)=1/x$, with $U=(0,1)$.
③ if $K\subset U$ is compact and U is open, then $d(K,\partial U)=\inf\{|x-y| \mid x\in K, y\in\mathbb{R}^n\setminus U\}>0$.

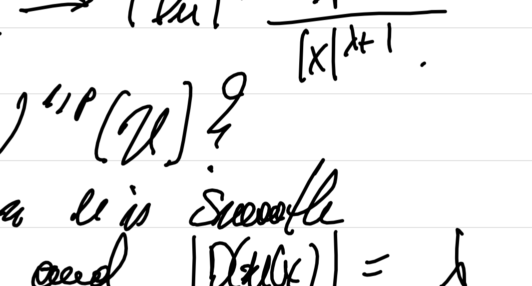


3.3 Weak Derivatives

"Notion of differentiability for L^p"
Def: Suppose $u,v\in L^1_{loc}(U)$ and α a multi-index. We say v is the α^{th} weak derivative of u if $\int_U uD^\alpha\phi\,dx=(-1)^{|\alpha|}\int_U v\phi\,dx$ $\forall\phi\in C_0^\infty(U)$.
Remarks: ① $\operatorname{supp}(D^\alpha\phi)$ is compact. If $u\in L^1_{loc}(U)$, then LHS makes sense. Similarly RHS.
② u,v obey the correct IBP formula.
Example: $u(x)=|x|$ is not differentiable at $x=0$, but it is weakly diffᵉ with $v=\operatorname{sgn}(x)$.
Lemma: Suppose $v,\tilde{v}\in L^1_{loc}$ are both the α^{th} weak derivative of $u\in L^1_{loc}(U)$. Then, $v\equiv\tilde{v}$ a.e.
Proof: $\forall\phi\in C_0^\infty(U)$, we have:
$$\int_U v\phi\,dx=(-1)^{|\alpha|}\int_U uD^\alpha\phi=\int_U\tilde{v}\phi\,dx$$
$$\Rightarrow\int_U(v-\tilde{v})\phi\,dx=0 \text{ (for all test functions)}$$
$\Rightarrow(v-\tilde{v})=0$ a.e. (write $v=D^\alpha u$, for the α^{th} weak derivative)

Defⁿ: Sobolev space:
$$W^{k,p}(U):=\{u\in L^1_{loc}(U) \mid u\in L^p(U) \text{ and the weak derivatives } D^\alpha u \text{ exist } \forall|\alpha|\le k \text{ with } D^\alpha u\in L^p(U)\}$$
Sobolev norm $\|u\|_{W^{k,p}(U)}:=\begin{cases}\left(\sum_{|\alpha|\le k}\int_U|D^\alpha u|^p dx\right)^{1/p}, & 1\le p<\infty\\ \sum_{|\alpha|\le k}\operatorname*{ess\,sup}_U|D^\alpha u|, & p=\infty.\end{cases}$

When $p=2$, write $H^{k,2}(U)=W^{k,2}(U)$. ← Hilbert
We denote by $W_0^{k,p}(U)$ the completion of $C_c^\infty(U)$ in the $W^{k,p}(U)$-norm. (i.e. $u\in W_0^{k,p}\Leftrightarrow\exists u_m\in C_c^\infty(U)$ s.t. $\|u-u_m\|_{W^{k,p}(U)}\to 0$. Also, $H_0^k=W_0^{k,2}(U)$)
"$u=0$ on ∂U".



Example: $(n\ge 2,\ \lambda>0)$. Let $U=B_1(0)\subset\mathbb{R}^n$. set
$$u(x)=\begin{cases}|x|^{-\lambda}, & x\in B_1(0)\setminus\{0\}\\ 0, & x=0,\end{cases}$$
Check: $D_iu=-\frac{\lambda x_i}{|x|^{\lambda+2}}\to|Du|=\frac{\lambda}{|x|^{\lambda+1}}$.
When is $u\in W^{1,p}(U)$?
Outside the origin u is smooth and $D_iu=-\frac{\lambda x_i}{|x|^{\lambda+2}}$ and $|D(u(x))|=\frac{\lambda}{|x|^{\lambda+1}}$,
$u(x)\in L^1_{loc}(B_1(0))$ iff. $\int_{B_1(0)}|x|^{-\lambda}dx<\infty$.
Using polar coordinates, $\int_{B_1(0)}|x|^{-\lambda}dx$
$=\omega_{n-1}\int_{(0,1)}r^{-\lambda}\cdot r^{n-1}dr<\infty\Leftrightarrow n-1-\lambda>-1$
surface area of unit sphere $\Leftrightarrow\boxed{\lambda<n}$
Suppose u has a weak derivative v on $B_1(0)$. Then, for K compact $\subset B_1(0)$ s.t. $0\notin K$, $\exists$ W open s.t. $0\notin W\supset W\supset K\Rightarrow$ $\forall\phi\in C_c^\infty(V)$
$$\int_W u\phi_{x_i}dx=-\int_W u_{x_i}(x)\phi(x)dx.$$
$$=-\int_W v(x)\phi(x)dx.$$
$\Rightarrow$ on K $v\equiv u_{x_i}$ a.e on K by a mollification argument and $v_i=u_{x_i}$ a.e. on U (by compact exhaustion of $U=B_1(0)$)
Thus, if $u\in H^{1,p}(U)$, then $\int|Du|^p<\infty$
$$\Leftrightarrow\int_{B_1(0)}\frac{|\lambda|^p}{|x|^{(\lambda+1)p}}dx<\infty\Leftrightarrow(\lambda+1)p<n$$
$$\Leftrightarrow\lambda<\frac{n-p}{p}$$
The converse implication is handled by showing $|Du(x)|$ is integrable and using Green's Theorem: $\forall\varepsilon>0$ $(\varepsilon\in(0,1))$.
$$\int_{B_1(0)\setminus B_\varepsilon(0)}u\phi_{x_i}dx=-\int_{B_1(0)\setminus B_\varepsilon(0)}u_{x_i}\phi(x)dx+\int_{\partial B_\varepsilon(0)}u\phi\nu^i ds$$
above as $\varepsilon\to 0$, $\left|\int_{\partial B_\varepsilon(0)}u\phi\nu^i ds\right|\le\|\phi\|_\infty\left|\int_{\partial B_\varepsilon(0)}|x|^{-\lambda}\nu^i ds\right|$
$$\le\|\phi\|_\infty\varepsilon^{n-1-\lambda}\to 0$$
$\forall\phi\in C_c^\infty(B_1(0))$.

Analysis of PDE.
Lecture 7

Example Classes

Example: $U = B_1(0) \subset \mathbb{R}^n$, $n \geq 2$, $\lambda > 0$.

$$u(x) = \begin{cases} |x|^{-\lambda}, & x \in B_1(0) \setminus \{0\}. \\ 0, & x = 0 \end{cases}$$

$$\int_U \frac{1}{|x|^\lambda} dx = C \int_{(0,1)} r^{-\lambda} \cdot r^{n-1} dr < \infty \iff \lambda < n.$$

Also $u \in L^p \iff p\lambda < n \iff \lambda < n/p$.

Look at $\phi \in C_c^\infty(B_1(0) \setminus \{0\})$, if u has a weak derivative v, then

$$v_i = D_i u = -\frac{\lambda x_i}{|x|^{\lambda+2}} \text{ on } B_1(0) \setminus \{0\}$$

$$\longrightarrow |Du| = \frac{|\lambda|}{|x|^{\lambda+1}} \longrightarrow v_i \in L^1_{loc}(U) \iff \lambda + 1 < n.$$

$\Rightarrow$ Assume $\lambda + 1 < n$. <u>Claim:</u> $v_i = \begin{cases} -\frac{\lambda x_i}{|x|^{\lambda+2}}, & x \neq 0. \\ 0, & x = 0. \end{cases}$

is the weak derivative of u in U. For $\phi \in C_c^\infty(U)$ by Stokes theorem

$$(-1)\int_{U \setminus B_\varepsilon(0)} u \cdot \phi_{x_i} dx = \int_{U \setminus B_\varepsilon(0)} D_i u \cdot \phi \, dx - \int_{\partial B_\varepsilon(0)} u \cdot \phi \, \vec{n} \cdot d\vec{S}$$

$$\left| \int_{\partial B_\varepsilon(0)} u \cdot \phi \, \vec{n} \, d\vec{S} \right| \leq \|\phi\|_\infty \cdot \int_{\partial B_\varepsilon} \varepsilon^{-\lambda} \vec{n} \cdot d\vec{S}$$

$$(\leq C) \cdot \varepsilon^{n-1-\lambda} \to 0, \ \varepsilon \to 0 \ (\lambda + 1 < n).$$

$\lesssim C$

$$\Longrightarrow - \int_U u \, \phi_{x_i} dx = \int_U v_i \, \phi \, dx$$

Remarks: (1) weak derivatives exist even if u is not continuous.

(2) Also, $D_i u \in L^p(U) \iff p(\lambda + 1) < n$.

$\Longrightarrow u \in W^{1,p}(U) \iff \lambda < \frac{n-p}{p}$

$\Longrightarrow$ if $p > n$ then $\lambda < 0$ and $u \in C^0(U)$

$\Longrightarrow$ larger $p \Rightarrow$ nicer functions.

Theorem: $\left(W^{k,p}(U), \|\cdot\|_{W^{k,p}(U)}\right)$ is a Banach space $k \in \{0, 1, 2, \dots\}$. $1 \leq p \leq \infty$.

Proof: (1) <u>Normed space</u>: straightforward. To prove the Δ-inequality, use Minkowski ineq. $\left(\sum (a_i + b_i)^p\right)^{1/p} \leq \left(\sum |a_i|^p\right)^{1/p} + \left(\sum |b_i|^p\right)^{1/p}$.

(2) <u>Completeness</u>: let u_j be a Cauchy sequence in $W^{k,p}(U)$.

Aim: $u_j \longrightarrow u$ in $W^{k,p}(U)$ for some $u \in W^{k,p}(U)$.

Note $\|D^\alpha v\|_{L^p(U)} \leq \|v\|_{W^{k,p}(U)}$ for $|\alpha| \leq k$. If set $v = u_j$, $\Rightarrow (D^\alpha u_j)_j$ is Cauchy in $L^p(U)$. By Completeness of $L^p(U)$, $\exists$ $u^\alpha \in L^p(U)$ s.t. $D^\alpha u_j \to u^\alpha$ in L^p for each $|\alpha| \leq k$. Call $u = \lim u^{(0,\dots,0)}$. <u>Claim:</u> u^α is the weak derivative $D^\alpha u$ of the limit u, i.e. $D^\alpha u$ exists and $D^\alpha u = u^\alpha$.

Let $\phi \in C_c^\infty(U)$. Since $u_j \in W^{k,p}(U)$, know $D^\alpha u_j$ exists and

$$(\forall j): \quad (-1)^{|\alpha|} \int_U u_j \, D^\alpha \phi \, dx = \int_U D^\alpha u_j \, \phi(x) \, dx.$$

By taking $j \to \infty$, using that $u_j \xrightarrow{L^p} u$, $D^\alpha u_j \xrightarrow{L^p} u^\alpha$, we get (Dom. Conv.)

$$(-1)^{|\alpha|} \int_U u \, D^\alpha \phi \, dx = \int_U u^\alpha \cdot \phi(x) \, dx$$

$$\Rightarrow D^\alpha u = u^\alpha \in L^p(U) \Rightarrow u \in W^{k,p}(U) \quad \square$$

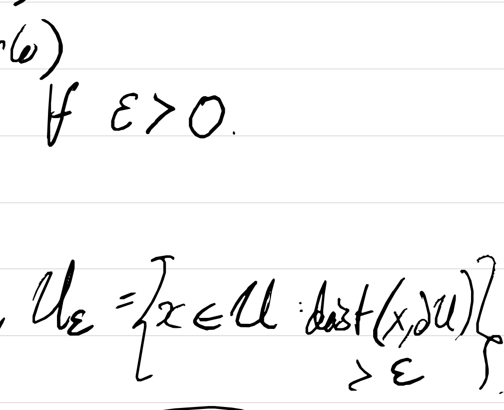
Approximations of Sobolev spaces

Convolution & mollification:

Def$^{\underline{n}}$ let $\eta(x) = \begin{cases} C e^{\left(-\frac{1}{1-|x|^2}\right)}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1. \end{cases}$

where C chosen s.t. $\int_{\mathbb{R}^n} \eta(x) dx = 1$.

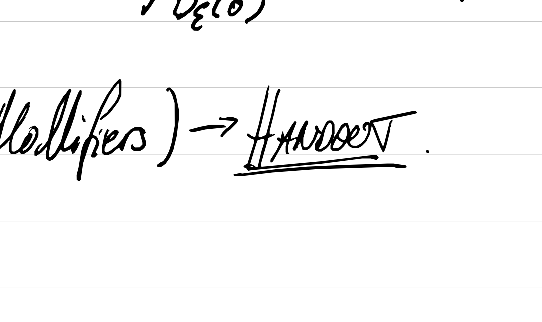
For each $\varepsilon > 0$ let $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \cdot \eta\left(\frac{x}{\varepsilon}\right)$. Called the standard mollifier.



Exercise:
- $\eta, \eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$
- $\text{supp}(\eta_\varepsilon) \in B_\varepsilon(0)$
- $\int \eta_\varepsilon(x) dx = 1 \ \forall \varepsilon > 0$.

Def$^{\text{n}}$: Given $U \subset \mathbb{R}^n$ open, $U_\varepsilon = \{x \in U : \text{dist}(x, \partial U) > \varepsilon\}$.

Given $f \in L^1_{loc}(U)$, the <u>mollification of f</u> is



$f_\varepsilon : \underbrace{U_\varepsilon}_{\text{not } U} \to \mathbb{R}$, by $f_\varepsilon(x) = \eta_\varepsilon * f$.

$$f_\varepsilon(x) = \int_U f(y) \, \eta_\varepsilon(x - y) \, dy = \int_{B_\varepsilon(0)} f(x-y) \, \eta_\varepsilon(y) \, dy.$$

Theorem: (Properties of Mollifiers) $\to$ <u>HANDOUT</u>.

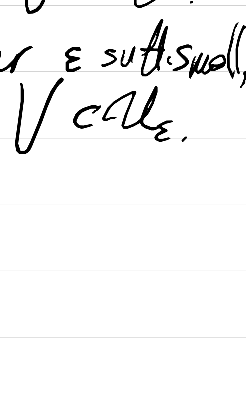
Let $f \in L^1_{loc}(U)$.
- (i) $f_\varepsilon \in C^\infty(U_\varepsilon)$
- (ii) $f_\varepsilon \to f$ a.e. in U as $\varepsilon \to 0$.
- (iii) if $f \in C(U)$, then $f_\varepsilon \to f$ uniformly on compact subsets of U.
- (iv) if $1 \leq p < \infty$ and $f \in L^p_{loc}(U)$ then $f_\varepsilon \to f$ in $L^p_{loc}(U)$, i.e. $\|f_\varepsilon - f\|_{L^p(V)} \to 0$ $\forall V \subset\subset U$.

<u>Key:</u> $f \in L^1_{loc}(U) \longrightarrow f_\varepsilon \in C^\infty$ is big improvement.

Lemma: (Local smooth approximation of Sobolev functions away from ∂U).

Let $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Set $u_\varepsilon = \eta_\varepsilon * u$ in U_ε. Then (i) $u_\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$.

(ii) $u_\varepsilon \to u$ in $W^{k,p}_{loc}(U)$.



Proof: (i) handout.

(ii) <u>Claim</u>: $D^\alpha u_\varepsilon = D^\alpha(u * \eta_\varepsilon)$ $= \eta_\varepsilon * D^\alpha u$ in U_ε, $\forall |\alpha| \leq k$.

$V \subset\subset U$. for ε suff. small, $V \subset U_\varepsilon$.

Since $u_\varepsilon \in C^\infty$, we can compute the classical derivative:

$$D_x^\alpha u_\varepsilon(x) = D_x^\alpha \int_U \eta_\varepsilon(x - y) u(y) \, dy$$

$$= \int_U D_x^\alpha \eta_\varepsilon(x - y) u(y) \, dy.$$

$$= (-1)^{+|\alpha|} \int_U \left(D_y^\alpha \eta_\varepsilon(x - y)\right) u(y) \, dy = (-1)^{2|\alpha|} \int_U \eta_\varepsilon(x - y) \, D_y^\alpha u(y) \, dy$$

using $\eta_\varepsilon(x - \cdot)$ are $C_c^\infty(U)$ for fixed $x \in U_\varepsilon$.

$$\dots = (\eta_\varepsilon * D^\alpha u)(x).$$

Next, fix $V \subset\subset U$. By theorem ($\to$) (iv) since $D^\alpha u \in L^p(U)$, then $D^\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u \longrightarrow D^\alpha u$ in $L^p(V)$ as $\varepsilon \to 0$.

$\Rightarrow \forall V \subset\subset U$, $\forall \delta > 0$ $\exists$ $\varepsilon_0 = \varepsilon_0(\delta, V)$ st

$$\|u_\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} \|D^\alpha u_\varepsilon - D^\alpha u\|^p_{L^p(V)} \leq \delta$$

$\forall \varepsilon \in (0, \varepsilon_0)$. $\square$

<u>Conclusion:</u> $u \in W^{k,p}(U)$ can be approximated by C^∞ functions away from ∂U.

ANALYSIS OF PDE
LECTURE 8

Lemma 8.3 if $u \in W^{k,p}(U)$, $1 \le p < \infty$ then $u_\varepsilon \to u$ in $W^{k,p}_{loc}(U)$ where $u_\varepsilon = \eta_\varepsilon * u$.

$$f_\varepsilon(x) = \int_{B_\varepsilon(0)} \eta_\varepsilon(y) f(x-y)\,dy$$

Ex: to drop

Theorem Suppose $U \subset \mathbb{R}^n$ is open + bounded and suppose $u \in W^{k,p}(U)$ for $1 \le p < \infty$. then $\exists (v_j) \in C^\infty(U) \cap W^{k,p}(U)$ s.t. $v_j \to u$ in $W^{k,p}(U)$. (We don't claim $v_j \in C^\infty(\bar{U})$)

Proof: (1) We have $U = \bigcup^\infty U_j$, where $U_j = \{x \in U \mid dist(x, \partial U) > 1/j\}$. Write $V_j = U_{j+3} \setminus \bar{U}_{j+1} \subset\subset U$ (use U is bounded)

Choose $V_0 \subset\subset U$ s.t. $U = \bigcup_{j=0}^\infty V_j$. Let $(\xi_j)_{j=0}^\infty$ be a partition of unity subordinate to V_j i.e.,
- $0 \le \xi_j \le 1$,
- $\xi_j \in C_c^\infty(V_j)$,
- $\sum_{j=1}^\infty \xi_j(x) = 1$ for $x \in U$.

Given $u \in W^{k,p}(U)$, (Ex) see that $\xi_j u \in W^{k,p}(U)$ and $supp(\xi_j u) \subset V_j$.

(2) "Smooth out our split-up function". Let $W_j = U_{j+4} \setminus \bar{U}_j \supset V_j$. Let

$u_j = \eta_{\varepsilon_j} * (\xi_j u)$. Fix $\delta > 0$. For each $j \ge 1$, we can choose ε_j sufficiently small s.t. $supp(u_j) \subset W_j$.
By Lemma 8.3 (typed notes), have $u_j \to \xi_j u$ in $W^{k,p}(W_j)$.

$$\|u_j - \xi_j u\|_{W^{k,p}(U)} = \|u_j - \xi_j u\|_{W^{k,p}(W_j)} \le \frac{\delta}{2^{j+1}}$$

(3) "Sum up everything together." Let $v = \sum_{j=0}^\infty u_j$. Note $u_j \neq 0$ on finitely many W_j's. So $v \in C^\infty(U)$ in any open nbhd as the sum is a finite sum of smooth functions. Also,

$$u(x) \cdot 1 = \sum_{j=0}^\infty \xi_j \cdot u \text{ on } U$$

So for any $V \subset\subset U$ we have:

$$\|v - u\|_{W^{k,p}(V)} \le \sum_{j=0}^\infty \|u_j - \xi_j u\|_{W^{k,p}(U)}$$

$$\le \delta \cdot \sum_{j=0}^\infty 2^{-(j+1)} = \delta.$$ Take supremum over $V \subset\subset U$, we have $\|v - u\|_{W^{k,p}(U)} \le \delta$ $\square$

Q. Can we approximate $W^{k,p}(U)$ by $u \in C^\infty(\bar{U})$

The boundary could be a problem: Cantor set C on $[0,1] \times \{0\}$ is closed in $\mathbb{R}^2$. If $U = \mathbb{R}^2 \setminus C$ is open but $\partial U = C$, very nasty.

Definition: Suppose $U \subset \mathbb{R}^n$ is bounded & open. Then we say ∂U is a $C^{k,\delta}$-domain if for every $p \in \partial U$ $\exists$ $r > 0$ and a function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\gamma \in C^{k,\delta}(\mathbb{R}^{n-1})$ and such that (after re-labelling axes)

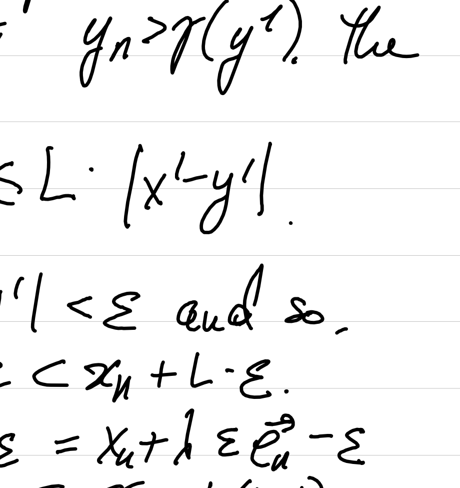
$$U \cap B_r(p) = \{(x', x_n) \in B_r(p) \mid x_n > \gamma(x')\}$$

$x' = (x_1, \dots, x_{n-1})$.

Theorem: Let $U \subset \mathbb{R}^n$ be open, bounded and ∂U be a $C^{0,1}$ domain (i.e. Lipschitz). Let $u \in W^{k,p}(U)$, some $1 \le p < \infty$. Then $\exists (u_j)_j \in C^\infty(\bar{U})$ s.t. $u_j \to u$ in $W^{k,p}(U)$.

Proof: (1) Fix $x_0 \in \partial U$. Since ∂U is Lipschitz, $\exists$ $r > 0$ and γ a Lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ s.t. $U \cap B_r(x_0) = \{x \in B_r(p) \mid x_n > \gamma(x')\}$.

Set $V = U \cap B_{r/2}(x_0)$



(2) Define the shifted point $x^\varepsilon = x + \lambda \varepsilon \vec{e}_n$ for $x \in V$ and $\varepsilon > 0$.

Claim: for large enough fixed $\lambda > 0$, $B_\varepsilon(x^\varepsilon) \subset U \cap B_r(x_0)$. i.e. NTP for $y \in B_\varepsilon(x^\varepsilon)$ that $y_n > \gamma(y')$. The Lipschitz condition:

$$|\gamma(x') - \gamma(y')| \le L \cdot |x' - y'|$$

we have $|y' - (x^\varepsilon)'| = |x' - y'| < \varepsilon$ and so,

$$\gamma(y') \le \gamma(x') + L\varepsilon < x_n + L \cdot \varepsilon.$$

By rearranging, $y_n > x_n^\varepsilon - \varepsilon = x_n + \lambda \varepsilon \vec{e}_n - \varepsilon = x_n + (\lambda - 1)\varepsilon$.

$\Rightarrow y_n > \gamma(y')$ if $\lambda \ge L + 1$.

Define $u_\varepsilon(x) = u(x^\varepsilon)$ for $x \in V$ (i.e. translation), set $v_{\delta,\varepsilon} = \eta_\delta * u_\varepsilon$ for $0 < \delta < \varepsilon$. Then $v_{\delta,\varepsilon} \in C^\infty(\bar{V})$. We have shown that $y^\varepsilon \in U \cap B_r(x_0)$, for $y \in V_\varepsilon$. Then $u^\varepsilon \in W^{k,p}(V_\varepsilon) \Rightarrow v_{\delta,\varepsilon} \in C^\infty(\bar{V})$.
Fix $\mu > 0$ small. Then we note

$$\|v_{\delta,\varepsilon} - u\|_{W^{k,p}(U)} \le \|v_{\delta,\varepsilon} - u_\varepsilon\|_{W^{k,p}(U)} + \underbrace{\|u_\varepsilon - u\|_{W^{k,p}(U)}}$$

the translation operator is cont$^{\underline{s}}$ in L^p norm.
We can pick $\varepsilon > 0$ s.t. (8) $\le \mu$. Fix $\varepsilon > 0$, pick $\delta < \varepsilon$ s.t. (1) $\le \mu$ (same proof as Lemma 8.3).

(3) Let x_0 vary over ∂U, see that the V's cover ∂U. Since ∂U is compact, we can find finitely many points $x_i \in \partial U$ and radii $r_i > 0$ with $V_i = U \cap B_{r_i/2}(x_i)$, $1 \le i \le N$. Choose $V_0 \subset\subset U$ s.t. $U = \bigcup_{i=0}^N V_i$.
By (2) we found $v_i \in C^\infty(\bar{V})$ s.t. $\|v_i - u\|_{W^{k,p}(V_i)} \le \mu$. By Lemma 8.3 $\exists$

$v_0 \in C^\infty(\bar{V}_0)$ s.t. $\|v_0 - u\|_{W^{k,p}(V_0)} \le \mu$.

(4) Let $(\xi_i)_{i=0}^N$ be a smooth partition of unity subordinate to the $\{V_0, \dots, V_N\}$. Define $v = \sum_{i=0}^N v_i \xi_i$. Then, $v \in C^\infty(\bar{U})$ and for all $|\alpha| \le k$

$$\|D^\alpha v - D^\alpha u\|_{L^p(U)}$$

$$\le \sum_{i=0}^N \|D^\alpha(\xi_i(v_i - u))\|_{L^p(V_i)}$$

$$\le C_k \sum_{i=0}^N \|v_i - u\|_{W^{k,p}(V_i)}$$

$$\le C_k (1+N)\mu = C\mu \to 0, \mu \to 0 \quad \square$$

Analyses Of PDE

Lecture 9

Recap: $\mathcal{U} \subset \mathbb{R}^n$. $C^\infty(\mathcal{U})$ = smooth functions. i.e. all derivatives continuous.
$C^\infty(\overline{\mathcal{U}})$ = all derivatives bounded & uniformly continuous.
$W^{k,p}(\mathcal{U}) = L^1_{loc}(\mathcal{U})$ functions with weak derivatives up to order k and in $L^p(\mathcal{U})$.

Example:

1. $|x| \notin C^\infty(-1,1)$ but $|x| \in W^{1,1}(-1,1)$
2. $\frac{1}{x} \in C^\infty(0,1)$, $\frac{1}{x} \notin C^\infty(\overline{0,1})$, $\frac{1}{x} \notin W^{1,1}(0,1)$.
3. $\frac{1}{x^2} \notin C^\infty(\overline{0,1})$, but $\frac{1}{x^2} \notin W^{1,1}(0,1)$.

Suppose $\mathcal{U}$ is bounded and $p \in [1,\infty)$.

1. $u \in W^{k,p}(\mathcal{U})$ is approx. by $u_\varepsilon \in C^\infty(\mathcal{U}_\varepsilon)$ in $W^{k,p}_{loc}(\mathcal{U})$.
2. $X = C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U})$ is dense in $W^{k,p}(\mathcal{U})$.
3. For good $\partial\mathcal{U}$, $X = C^\infty(\overline{\mathcal{U}})$ is dense in $W^{k,p}(\mathcal{U})$.

Extensions and Traces:

Suppose $u \in W^{k,p}(\mathcal{U})$, $\mathcal{U} \subset \mathbb{R}^n$ open and bounded. Can we extend $u \to \overline{u}$ defined on $\mathbb{R}^n$?

$$\overline{u} = \begin{cases} u & \text{on } \mathcal{U} \\ 0 & \text{on } \mathcal{U}^c \end{cases}$$

At most, we expect $\overline{u} \in W^{k,p}(\mathbb{R}^n)$.

Theorem 3.5: (Calderon '61, Stein '70).
Assume $\mathcal{U}$ is bounded and $\partial\mathcal{U}$ is C^1. Choose V bounded in $\mathbb{R}^n$ s.t. $\mathcal{U} \subset\subset V$, and let $1 \le p < \infty$. Then $\exists$ bounded linear operator

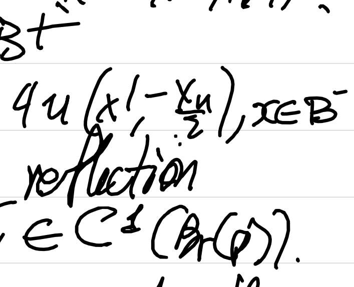
$$E: W^{1,p}(\mathcal{U}) \to W^{1,p}(\mathbb{R}^n)$$
$$u \mapsto E(u) = \overline{u} \text{ s.t.}$$

for all $u \in W^{1,p}(\mathcal{U})$.

(i) $\overline{u}|_{\mathcal{U}} = u$ a.e.
(ii) $\operatorname{supp}(E(u)) \subset V$
(iii) $\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \le C \cdot \|u\|_{W^{1,p}(\mathcal{U})}$, where $C = C(\mathcal{U}, V, p)$.

Eu is the *extension* of u to $\mathbb{R}^n$.

Proof: ① Fix $p \in \partial\mathcal{U}$ and suppose that $\partial\mathcal{U}$ is flat near p. So we assume $\exists r>0$ s.t. $B^+ = B_r(p) \cap \{x_n \ge 0\} \subset \overline{\mathcal{U}}$



$B_- = B_r(p) \cap \{x_n < 0\} \subset \mathbb{R}^n \setminus \overline{\mathcal{U}}$

Suppose also $u \in C^1(\overline{\mathcal{U}})$. Denote $x' = (x_1, \dots, x_{n-1})$.
Denote

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x', -x_n) + 4u(x', -\frac{x_n}{2}), & x \in B^- \end{cases}$$

Called a higher-order reflection of u from B^+ to B^-. Claim: $\overline{u} \in C^1(B_r(p))$.
Clearly, $\overline{u} \in C^0(B_r(p))$. We compute the derivatives:

$$\partial_{x_n} \overline{u}(x) = \begin{cases} \partial_{x_n} u(x), & x \in B_+ \\ 3\partial_{x_n} u(x', -x_n) - 2\partial_{x_n} u(x', -\frac{x_n}{2}), & x \in B^- \end{cases}$$

$$\Rightarrow \partial_{x_n} \overline{u}\Big|_{x_n = 0^+} = \partial_{x_n} \overline{u}\Big|_{x_n = 0^-}$$

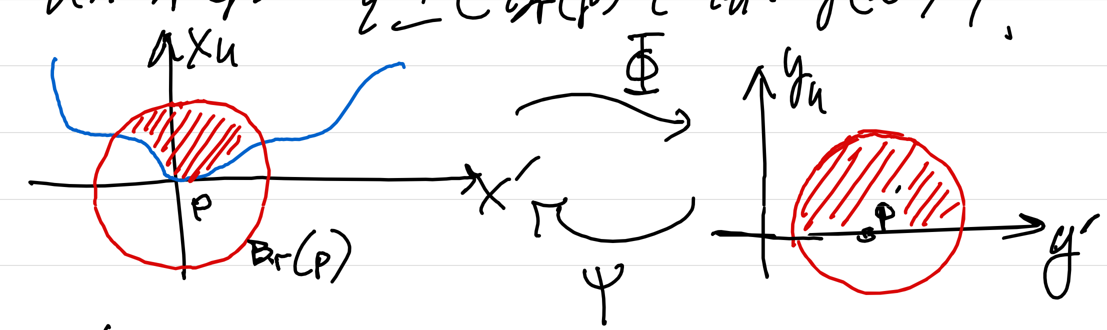
Also

$$\partial_{x_i} \overline{u} = \begin{cases} \partial_{x_i} u, & x \in B^+ \\ -3\partial_{x_i} u(x', -x_n) + 4\partial_{x_i} u(x', -\frac{x_n}{2}), & x \in B^- \end{cases}$$

$$\Rightarrow D^\alpha \overline{u}\Big|_{x_n = 0^+} = D^\alpha \overline{u}\Big|_{x_n = 0^-} \quad \forall |\alpha| \le 1.$$

Can also show that $\|\overline{u}\|_{W^{1,p}(B_r(p))} \le C\|u\|_{W^{1,p}(B^+)}$ with C independent of u. (check!)
Done in case ① by $E(u) = \overline{u}$.

③ Suppose $\partial\mathcal{U}$ not flat near p. Since $\partial\mathcal{U}$ is C^1 $\exists r > 0$ and $\gamma: \mathbb{R}^{n-1} \to \mathbb{R}$ s.t. $\mathcal{U} \cap B_r(p) = \{x \in B_r(p) \mid x_n > \gamma(x')\}$.



Define $\Phi: \mathbb{R}^n \to \mathbb{R}^n$, $\Phi(x) = y$ given by $y_i = x_i$ if $i = 1, \dots, n-1$ and $y_n = x_n - \gamma(x')$.
So $\Phi: \partial\mathcal{U} \to \{y_n = 0\}$. Note Φ is invertible.
$\Psi = \Phi^{-1}$, $\Psi(y) = x$, is given by $\begin{cases} x_i = y_i, & i = 1, \dots, n-1 \\ x_n = y_n + \gamma(y'). \end{cases}$
Check that $\Psi \circ \Phi = \Phi \circ \Psi = \mathrm{Id}_{\mathbb{R}^n}$.
$\Phi(\mathcal{U} \cap B_r(p)) \subset \{y_n > 0\}$ and both are C^1 with $\det D\Phi = \det D\Psi = 1$.
$\Rightarrow \Phi$ is a C^1 diffeo. About p, $\exists$ open set W s.t. $\Phi(W) = B_\delta(\tilde{p})$, some $\delta > 0$. $\Phi(p) = \tilde{p}$.
$\Phi(\mathcal{U} \cap W) = B_\delta(\tilde{p}) \cap \{y_n \ge 0\} = B_+$.
Define $v(y) = u(\Psi(y))$ for $y \in B_+$. Then $v \in C^1(B_+)$ and by ② $\exists$ extension $\overline{v}(y) \in C^1(B_\delta(\tilde{p}))$ s.t. $\overline{v}|_{B_+} = v$ and

$$\|\overline{v}\|_{W^{1,p}(\mathbb{R}^n)} \le C \cdot \|v\|_{W^{1,p}(B_+)}.$$

Define $\overline{u}(x) = \overline{v}(\Phi(x))$, then $\overline{u} \in C^1(W)$ and $\|\overline{u}\|_{W^{1,p}(W)} \underset{\text{Sheet 2}}{\le} C \cdot \|u\|_{W^{1,p}(\mathcal{U})}$.

④ Now have local extensions $\forall p \in \partial\mathcal{U}$ to W. Let $\{W_0, \dots, W_N\}$ be a finite subcover of $\mathcal{U}$.
$\Rightarrow \mathcal{U} \subset \bigcup_{i=0}^N W_i$ with extensions $\overline{u}_i \in C^1(W_i)$.
Let $(\xi_i)_{i=0}^N$ be a partition of unity subordinate to $\{W_i\}$ $\Rightarrow \operatorname{supp}(\xi_i) \subset\subset W_i$ and $\sum_{i=0}^N \xi_i = 1$ on $\mathcal{U}$. Let $\overline{u} = \sum_{i=0}^N \xi_i \overline{u}_i$ where $\overline{u}_0 = u$. Then $\overline{u}|_{\mathcal{U}} = u$ a.e. $\overline{u} \in C^1_c(\mathbb{R}^n)$ and $\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C \cdot \|u\|_{W^{1,p}(\mathcal{U})}$.

May assume $\operatorname{supp}(\overline{u}) \subset V$, some $\mathcal{U} \subset\subset V$ by X by some cut off function ($\mathcal{U} \subset\subset S \subset\subset V$, $X|_{\mathcal{U}} = 1$, $X|_{S^c} = 0$)

⑥ Given $u \in W^{1,p}(\mathcal{U})$ by Th. 3.4. $\exists (u_j) \subset C^\infty(\overline{\mathcal{U}})$ s.t. $u_j \to u$ in $W^{1,p}(\mathcal{U})$.
Claim: $(E(u_j))_j$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$.
Since $u_j \in C^\infty(\overline{\mathcal{U}}) \subset C^1(\overline{\mathcal{U}})$, by previous steps, $E u_j \in W^{1,p}(\mathbb{R}^n)$. By linearity,

$$\|E(u_j) - E(u_k)\|_{W^{1,p}(\mathbb{R}^n)} \overset{\text{linearity}}{=} \|E(u_j - u_k)\|_{W^{1,p}(\mathbb{R}^n)}$$
$$\le C \cdot \|u_j - u_k\|_{W^{1,p}(\mathcal{U})} \to 0$$

since $(u_j)_{j \in \mathbb{N}}$ is a convergent $\Rightarrow$ Cauchy in $W^{1,p}(\mathcal{U})$.
$\Rightarrow Eu = \lim_{j \to \infty} E(u_j)$ (and limit is independent of approximating sequence).

Remarks: for $\partial\mathcal{U} \in C^k$ get

$$E: W^{k,p}(\mathcal{U}) \to W^{k,p}(\mathbb{R}^n).$$

Given $u \in C^k(\overline{\mathcal{U}})$ set
(flat case)

$$\overline{u}(x) = \begin{cases} u(x), & x \in B^+ \\ \sum_{j=1}^n c_j u(x', -\frac{x_n}{j}), & x \in B_- \end{cases}$$

Ex: check for matching at boundary need $\sum_{j=1}^n c_j \left(-\frac{1}{j}\right)^m = 1$, for all $m = 0, \dots, k-1$.
(check!)

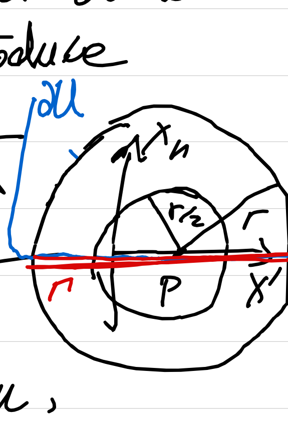
Analysis of PDE
Lecture 10

Traces if $u \in C^0(\bar{U}) \to u|_{\partial U}$ make sense.
if $u \in W^{k,p}(U) \longrightarrow u|_{\partial U}$?

Theorem: let $U \subset \mathbb{R}^n$ be open, bounded and ∂U is C^1. Then $\exists$ a bounded linear operator
$$T: W^{1,p}(U) \longrightarrow L^p(\partial U)$$
called the trace of u on ∂U, s.t.
(i) $T(u) = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C^0(\bar{U})$.
(ii) $\|T(u)\|_{L^p(\partial U)} \le c \cdot \|u\|_{W^{1,p}(U)}$ $\forall u \in W^{1,p}(U)$.
$c = C(U,p)$

Remark we have $u, Du \in L^p$.
$\Rightarrow$ control of u on ∂U.

Proof: (Sketch)
(1) Suppose $u \in C^1(\bar{U})$ and ∂U is flat near some point $p \in \partial U$. Introduce
$B_+ = B_r(p) \cap \{x_n \ge 0\} \subset \bar{U}$
$B_- = B_r(p) \cap \{x_n < 0\} \subset \mathbb{R}^n \setminus \bar{U}$
Γ = portion of ∂U within $B_{r/2}(p)$.



Pick $\xi \in C_c^\infty(B_r(p))$ s.t. $0 \le \xi \le 1$ on $B_r(p)$ and $\xi = 1$ on $B_{r/2}(p)$. Then,
$$\int_\Gamma |u(x', 0)|^p dx' \le \int_{B_r(p) \cap \{x_n = 0\}} \xi |u(x', 0)|^p dx'$$
(FTC)
$$= (-1) \int_{B_+} \partial_{x_n}(\xi \cdot |u|^p) dx_n dx'$$
(Sheet 2)
$$= (-1) \int_{B_+} |u|^p \partial_{x_n}\xi + p|u|^{p-1}\operatorname{sgn}(u)\partial_{x_n}u \cdot \xi \, dx$$
$$\le C_p \left(\int_{B_+} |u|^p + |Du|^p dx \right)$$ (Recall Young's ineq.: $|ab| \le \frac{|a|^m}{m} + \frac{|b|^n}{n}$, $\frac{1}{m} + \frac{1}{n} = 1$; $m = \frac{p}{1-p}, n = p$).
$$\le C_p \cdot \|u\|^p_{W^{1,p}(U)}.$$

In sheet 2, please do it :) (complete the proof).
$\rightarrow$ extend to general boundary and use ∂U is compact.
Defining the map $T(u) = u|_{\partial U}$ for each $u \in C^1(\bar{U})$ and you will have $\|T(u)\|_{L^p(\partial U)} \le C \cdot \|u\|_{W^{1,p}(U)}$, then conclude using $C^\infty(\bar{U})$ is dense in $W^{1,p}(U)$.

Remark: $W_0^{k,p}(U)$ is the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$-norm.
So, if $u \in W_0^{k,p}(U)$ then $\exists (u_j) \in C_c^\infty(U)$ with $u_j \to u$ in $W^{k,p}(U)$. $\Rightarrow T(u) = \lim_j T(u_j)$ (T is bdd linear $\Rightarrow$ cont.) $= \lim_j u_j|_{\partial U} = 0$.
In fact, the converse is true also.
$T(u) = 0 \Leftrightarrow u \in W_0^{k,p}(U)$.
(2) if $u \in W^{k,p}(U)$ then can define trace for $Du, \dots, D^{k-1}u$.

Sobolev inequalities: Trade differentiability (k) $\rightarrow$ for integrability (p).
" $\not\leftarrow$ "

E.g.: if $f' \in L^1(\mathbb{R})$ then $f \in L^\infty(\mathbb{R})$
but if $f \in L^\infty(\mathbb{R}) \not\Rightarrow f' \in L^1(\mathbb{R})$.

Idea: $\|u\|_{L^q(\mathbb{R}^n)} \le C \cdot \|Du\|_{L^p(\mathbb{R}^n)} (+\|u\|_{L^p})$.
$q > p$

three cases: (1) $1 \le p < n$, $(n \ge 2)$ (2) $p = n$, (3) $p \in (n, \infty]$.

Lemma: (3.4 in notes). let $n \ge 2$ and $f_1, \dots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. for any $1 \le i \le n$ denote $\tilde{x}_i = (x_1, \dots, \hat{x}_i, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. Set $f(x) = \prod_{i=1}^n f_i(\tilde{x}_i)$, function of n variables.
Then $f \in L^1(\mathbb{R}^n)$ with $\|f\|_{L^1(\mathbb{R}^n)} \le \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$.

Proof: We use induction. Case $n=2$:
$f(x) = f_1(x_1) \cdot f_2(x_2)$.
$$\|f\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |f_1(x_1)||f_2(x_2)| \overset{\text{Fubini}}{=} \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})}$$
Suppose n true, WTP for $n+1$:
Write $f(x) = (f_1(\tilde{x}_1) \times \dots \times f_n(\tilde{x}_n)) f_{n+1}(\tilde{x}_{n+1})$
$= F(x) \cdot f_{n+1}(\tilde{x}_{n+1})$.
Fix x_{n+1} and integrate over $x_1, \dots, x_n$:
$$\int_{\mathbb{R}^n} |f(\xi_1, \dots, \xi_n, x_{n+1})| d\xi_1 \times \dots \times d\xi_n$$
$$= \int_{\mathbb{R}^n} \underbrace{F(\xi, x_{n+1})}_{L^q} \underbrace{f_{n+1}(\xi)}_{L^p} d^n\xi, \quad p = n, \; q = \frac{n}{n-1}, \; n \ge 2$$
Hölder
$$\le \|F(\cdot, x_{n+1})\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \times \|f_{n+1}\|_{L^n(\mathbb{R}^n)}.$$
Apply induction hyp. to
$$\|F(\cdot, x_{n+1})\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = \|F(\cdot, x_{n+1})^{\frac{n}{n-1}}\|^{\frac{n-1}{n}}_{L^1(\mathbb{R}^n)}$$
$$\le \prod_{i=1}^n \|f_i(\cdot, x_{n+1})^{\frac{n}{n-1}}\|^{\frac{n-1}{n}}_{L^{n-1}(\mathbb{R}^{n-1})}$$
$$= \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}.$$
Integrate over x_{n+1},
$$\|f\|_{L^1(\mathbb{R}^{n+1})} \le \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \int_{\mathbb{R}} \prod_{i=1}^n \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})} dx_{n+1}$$
Generalized Hölder:
$$\left\| \prod_{i=1}^n f_i \right\|_{L^1} \le \prod_{i=1}^n \|f_i\|_{L^{p_i}} : \sum \frac{1}{p_i} = 1.$$
$$\underset{\text{with } p_i = n}{\le} (\text{G.H}) \quad \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^n \left(\int_{\mathbb{R}} \|f_i(\cdot, x_{n+1})\|^n_{L^n(\mathbb{R}^{n-1})} dx_{n+1} \right)^{1/n}$$
$$= \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \cdot \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R}^n)} = \prod_{i=1}^{n} \|f_i\|_{L^n(\mathbb{R}^n)} \checkmark$$

Theorem (Gagliardo–Nirenberg–Sobolev (GNS) inequality [59]).
Assume $1 \le p < n$ (valid when $n \ge 2$). Then, $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$, where $p^* = \frac{np}{n-p}$, is the Sobolev conjugate to p. ($\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$). Moreover, the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is cont$^{\underline{s}}$ i.e. $\exists$ $C = C(n,p) > 0$ s.t. $\forall u \in W^{1,p}(\mathbb{R}^n)$,
$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \cdot \|Du\|_{L^p(\mathbb{R}^n)}.$$

Remarks: (1) $p^* > p$: (2) nothing is said about $\|Du\|_{L^{p^*}}$

Intuition: Consider $f: \mathbb{R}^n \to \mathbb{R}$, L^p measure width & height of function. Eg (1), $f_1 = A \cdot \mathbb{1}_W(x)$, then $\|f\|_{L^p} \sim |A| \cdot \operatorname{Vol}(W)^{1/p}$
(2) let $\phi \in C_c^\infty(\mathbb{R}^2)$. $f_2(x) = \phi(x) \cdot e^{i\vec{\omega} \cdot \vec{x}}$.
s.t. $\|f_2\|_\infty \le 1$, $\operatorname{supp}(f_2) \le C$ unif bdd in $\vec{\omega}$.
$\partial_1 f_2 = \phi' e^{i\vec{\omega} \cdot \vec{x}} + i\phi \underset{\hookrightarrow \text{can grow.}}{\omega_1} \cdot e^{i\vec{\omega} \cdot \vec{x}}$

$\rightarrow |Df_2| \not\le$ no uniform bound.
(3) $f_3(x) = |\omega|^{-k} \phi(x) e^{i\vec{\omega} \cdot \vec{x}}$, $k \ge 0$.
$\operatorname{freq}(f_3) \sim |\vec{\omega}|$ and $|D^l f_3| \le C$ uniform in ω if $|l| \le k$.
Use $W^{k,p}$ to measure width, height, frequency
(4) $f_4(x) = A\phi\left(\frac{x}{R}\right) \exp(i\vec{\omega} \cdot \vec{x})$.
$$\|f_4\|_{W^{1,p}} \sim \left(\int_{|x| \le R} |A \phi \cdot e^{i\omega x}|^p + \int_{|x| \le R} \left|\frac{A}{R}\phi' e^{i\omega x} + A\phi\omega e^{i\omega x}\right|^p \right)^{1/p}$$
$$\sim (|A|^p R^n |\omega|^p)^{1/p} \sim |A| \cdot |\omega| \cdot V^{1/p}$$
Uncertainty principle, $\delta_x \cdot \delta_p \gtrsim C > 0$.
volume $\times$ freq $\gtrsim C > 0$.
A function of frequency ω must be spread out on a ball of radius at least $\gtrsim \frac{1}{\omega}$ $\Longrightarrow$ support must have measure $\gtrsim \omega^{-n}$ $\Longrightarrow \omega \gtrsim \frac{1}{V^n}$.
$$\Rightarrow \|f\|_{W^{1,p}} \simeq |A| \cdot V^{1/p} \cdot |\omega| \gtrsim |A| \cdot V^{1/p - \frac{1}{n}} = |A| \cdot V^{\frac{1}{p^*}} \simeq \|f\|_{L^{p^*}}$$

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- If $u \equiv 1$, then (*) fails but of course if $u \in W^{1,p}(\mathbb{R}^n) \Rightarrow |u| \to 0$ as $|x| \to \infty$.
- Use density of $C_c^\infty(\mathbb{R}^n)$ in $W^{k,p}(\mathbb{R}^n) \cong W_0^{k,p}(\mathbb{R}^n)$.

<u>Proof (GNS):</u> ① Assume $u \in C_c^\infty(\mathbb{R}^n)$ and consider $p=1$. By FTC, and compact support, $u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i$, $i = 1, \dots, n$.

$$\Rightarrow |u(x)| \le \int_{\mathbb{R}} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, dy_i = f_i(\tilde{x}_i).$$

Then, $|u(x)|^n = |u| \times \dots \times |u| \le f_1(\tilde{x}_1) \times \dots \times f_n(\tilde{x}_n)$

$= \prod_{i=1}^n f_i(\tilde{x}_i)$. Integrate over $\mathbb{R}^n$.

$$\| |u|^{\frac{n}{n-1}} \|_{L^1(\mathbb{R}^n)} \le \Big\| \prod_{i=1}^n f_i(\tilde{x}_i)^{\frac{1}{n-1}} \Big\|_{L^1(\mathbb{R}^n)}$$

(Lemma 3.4)

$$\le \prod_{i=1}^n \| f_i^{\frac{1}{n-1}} \|_{L^{n-1}(\mathbb{R}^{n-1})} \overset{\text{Check!}}{=} \| Du \|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}.$$

$$\Rightarrow \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \|Du\|_{L^1(\mathbb{R}^n)}.$$

($p^* = \frac{n}{n-1}$ if $p=1$).

$C_c^\infty(\mathbb{R}^n)$ dense in $W^{1,1}(\mathbb{R}^n) \Longrightarrow$ result follows by density.

② Suppose $p > 1$. Consider $v(x) = |u(x)|^\gamma$, $\gamma > 1$ chosen later. Compute

$$Dv \overset{\text{(sheet 2)}}{=} \gamma \cdot \operatorname{sgn}(u) \cdot |u|^{\gamma-1} Du.$$

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{n-1}{n}} = \| |u|^\gamma \|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}$$

$$\overset{p=1 \text{ case}}{\le} \| D(|u|^\gamma) \|_{L^1(\mathbb{R}^n)}$$

$$= \| \gamma \cdot \operatorname{sgn}(u) \cdot |u|^{\gamma-1} Du \|_{L^1(\mathbb{R}^n)}$$

$$\le \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du|\, dx$$

$$\overset{\text{Hölder}}{\le} \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}} dx \right)^{1-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p dx \right)^{1/p}$$

$q = \frac{p}{p-1}$.

Choose γ s.t. $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} \Rightarrow \gamma = \frac{p(n-1)}{n-p} > 1$.

In particular, $\frac{\gamma n}{n-1} = \frac{np}{n-p} = p^*$, so we get

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\frac{n-1}{n}} \le \frac{p(n-1)}{n-p} \left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Longrightarrow \left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{1/p^*} \le \frac{p(n-1)}{n-p} \cdot \|Du\|_{L^p(\mathbb{R}^n)}.$$

$$\Rightarrow \|u\|_{L^{p^*}(\mathbb{R}^n)} \le \underbrace{\frac{p(n-1)}{n-p}}_{C} \cdot \|Du\|_{L^p(\mathbb{R}^n)}.$$

Note $C(n,p) \to \infty$ as $p \uparrow n$.

$\Rightarrow \|u\|_{L^{p^*}(\mathbb{R}^n)} \le C(n,p) \|u\|_{W^{1,p}(\mathbb{R}^n)}$ and conclude using density. □

<u>Corollary</u> (GNS for $U \subset \mathbb{R}^n$).

Suppose $U \subset \mathbb{R}^n$ is open and <u>bounded</u> with C^1 boundary. Let $1 \le p < n$. If $p^* = \frac{np}{n-p}$, then $W^{1,p}(U) \subset L^{p^*}(U)$ and $\exists\, C = C(\underline{U}, n, p)$, s.t.

$$\|u\|_{L^{p^*}(U)} \le C \|u\|_{W^{1,p}(U)}, \ \forall u \in W^{1,p}(U).$$

<u>Proof:</u> Exercise. Use extension theorem and GNS.

<u>Corollary:</u> (Poincaré inequality) Let $U \subset \mathbb{R}^n$ be open and bounded. Suppose $u \in W_0^{1,p}(U)$ for some $1 \le p < n$. Then $\exists\, C = C(p, q, n, U)$ s.t.

$$\|u\|_{L^q(U)} \le C \cdot \|Du\|_{L^p(U)} \text{ for each}$$

$1 \le q \le p^*$. In particular, as $1 \le p \le p^*$ (i.e. take $q = p$).

$$\|u\|_{L^p(U)} \le C \cdot \|Du\|_{L^p(U)}.$$

<u>Remarks:</u> ① on $W_0^{1,p}(U)$, we have:

$$\|u\|_{W^{1,p}(U)} \simeq \|Du\|_{L^p(U)}.$$

② really need $u \in W_0^{1,p}(U)$ to kill off constant functions.

<u>Proof:</u> Use that $W_0^{1,p}(U)$ is the closure of $C_c^\infty(U)$ under the $W^{1,p}$-norm. So $\exists\, (u_m) \subset C_c^\infty(U)$ s.t. $\|u_m - u\|_{W^{1,p}(U)} \to 0$. Since u_m vanish near ∂U we can extend u_m to zero on $\mathbb{R}^n \setminus U$ to get $\bar{u}_m \in C_c^\infty(\mathbb{R}^n)$. Apply the GNS inequality:

$$\|\bar{u}_m\|_{L^{p^*}(\mathbb{R}^n)} \le C \|D\bar{u}_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

Send $m \to \infty$ and use the fact that $\bar{u} = 0$ on $\mathbb{R}^n \setminus U$ to get

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du\|_{W^{1,p}(\mathbb{R}^n)}.$$

$\Rightarrow$ proves claim for $q = p^*$. Since $|U| < \infty$ by Hölder

$$\|u\|_{L^q(U)} \le C \cdot \|u\|_{L^{p^*}(U)} \le C \cdot \|Du\|_{L^p(U)}. \ □$$

<u>Case ②:</u> $p = n$, $p^* \to \infty$, so might expect $\|u\|_{L^\infty(U)} \le C \cdot \|u\|_{W^{1,n}}$. False for $n > 1$.

<u>Case ③:</u> $n < p < \infty$, "might expect better than L^∞", i.e. continuity.

<u>Theorem</u> : <u>Morrey's inequality</u>

Let $n < p < \infty$, then $\exists\, C = C(p,n) > 0$ s.t. $\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C \cdot \|u\|_{W^{1,p}(\mathbb{R}^n)}$ where $\gamma = 1 - \frac{n}{p} < 1$, i.e. such functions are (up to a.e. identification) Hölder continuous.

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\gamma}(\mathbb{R}^n).$$

(the above inequality is true for <u>all test functions</u>).

<u>Proof:</u> Let Q be an <u>open</u> cube of side length $r > 0$ and $\vec{0} \in Q$ and set

$$\bar{u} = \frac{1}{|Q|} \int_Q u(x)\, dx \ \ (\text{i.e. "}\bar{u} = \text{avg"}).$$

Then, $|\bar{u} - u(0)| \le \frac{1}{|Q|} \int |u(x) - u(0)|\, dx$. Since $u \in C_c^\infty(\mathbb{R}^n)$ by FTC:

$$u(x) - u(0) \overset{\text{FTC}}{=} \int_0^1 \frac{d}{dt}(u(tx))\, dt$$

$$\overset{\text{Chain rule}}{=} \sum_{i=0}^n \int_0^1 x^i \frac{\partial u}{\partial x^i}(tx)\, dt.$$

So $|u(x) - u(0)| \le r \sum_{i=1}^n \int_0^1 |\partial_{x_i} u(tx)|\, dt$ ($|x_i| < r$)

$$\Rightarrow |\bar{u} - u(0)| \le \frac{r}{|Q|} \int_Q \int_0^1 \sum_{i=1}^n |\partial_{x_i} u(tx)|\, dt\, dx$$

$$\overset{\substack{\text{Fubini \&} \\ \text{rescaling} \\ y = tx}}{=} \frac{r}{|Q|} \int_0^1 t^{-n} \left(\sum_{i=1}^n \int_{tQ} |\partial_{x_i} u(y)|\, dy \right) dt$$

$$\overset{\text{Hölder}}{\le} \frac{r}{|Q|} \int_0^1 t^{-n} \left(\sum_{i=1}^n \|\partial_{x_i} u\|_{L^p(tQ)} |tQ|^{1/q} \right) dt$$

$|Q| = r^n$

$$\le \frac{r}{r^n} \int_0^1 t^{-n} \|Du\|_{L^p(\mathbb{R}^n)} t^{n/q} r^{n/q}\, dt$$

$$= \left(\frac{r^{1-n/p}}{1 - n/p} \right) \times \|Du\|_{L^p(\mathbb{R}^n)}, \text{ i.e.}$$

$$|\bar{u} - u(0)| \le \frac{r^\gamma}{\gamma} \|Du\|_{L^p(\mathbb{R}^n)}.$$

By translation, $|\bar{u} - u(x)| \le \frac{r^\gamma}{\gamma} \cdot \|Du\|_{L^p(\mathbb{R}^n)}$ $\forall x \in Q$.

So by triangle inequality.

$$|u(x) - u(y)| \le |u(x) - \bar{u}| + |\bar{u} - u(y)|$$

$$\le 2 \frac{r^\gamma}{\gamma} \|Du\|_{L^p(\mathbb{R}^n)},$$

$\forall x, y \in Q$. Given any two points $x, y \in \mathbb{R}^n$, $\exists$ a cube Q of side length $r = 2|x-y|$, s.t. $r = 2|x-y|$ s.t. $x, y \in Q$.

$$\Rightarrow \frac{|u(x) - u(y)|}{|x-y|^\gamma} \le C \cdot \|Du\|_{L^p(\mathbb{R}^n)}.$$

taking sup,

$$\xrightarrow[x \neq y]{} [u]_{C^{0,\gamma}(\mathbb{R}^n)} \le C \cdot \|Du\|_{L^p(\mathbb{R}^n)}.$$

Finally, to control the $\sup |u(x)|$ note that any $x \in \mathbb{R}^n$ belongs to a cube of side length 1. So $|u(x)| \le |\bar{u}| + |u - \bar{u}|$

$$\le \int_Q |u(x)|\, dx + C \cdot \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\overset{\text{Hölder}}{\le} |Q|^{1/q} \cdot \|u\|_{L^p(\mathbb{R}^n)} + C \|Du\|_{L^p(\mathbb{R}^n)}$$

$\le C \cdot \|u\|_{W^{1,p}(\mathbb{R}^n)}$. Note C is independent of x and we finally obtain:

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C \cdot \|u\|_{W^{1,p}(U)}.$$

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<u>Corollary</u>: Suppose $u \in W^{1,p}(U)$ for $U \subset \mathbb{R}^n$ & $n < p \le \infty$ open, bounded set with $\partial U \in C^1$. Then, $\exists!$ $u^* \in C^{0,\gamma}(\mathbb{R}^n)$, $\gamma = 1 - n/p$ s.t. $u = u^*$ a.e. in U and

$$\|u^*\|_{C^{0,\gamma}(U)} \le C \cdot \|u\|_{W^{1,p}(U)}$$ where $C = C(p, n, U)$.

<u>Proof:</u> By the extension theorem, $\exists$ $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ s.t. $\bar{u} = u$ a.e. on U. Since $\bar{u}$ has compact support, by the approximation theorem, $\exists$ $(u_j) \subset C_c^\infty(\mathbb{R}^n)$ s.t. $u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$. Note Morrey's inequality $\|u_m - u_j\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C \cdot \|u_m - u_j\|_{W^{1,p}(\mathbb{R}^n)}$
$\Rightarrow (u_j)_j$ is Cauchy in the Banach space $C^{0,\gamma}(\mathbb{R}^n) \Rightarrow \exists$ $\bar{u}^* \in C^{0,\gamma}(\mathbb{R}^n)$ s.t. $u_j \to \bar{u}^*$ in $C^{0,\gamma}(\mathbb{R}^n)$. Then $u^* = \bar{u}^*|_U$ satisfies the conditions of the th^m. $\square$

<u>Summary:</u> if $U \subset \mathbb{R}^n$ is open, bdd with $\partial U \in C^1$.

if $1 \le p < n$, then $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ $(p^* > p)$

if $n < p < \infty$ then $W^{1,p}(U) \hookrightarrow C^{0,\gamma}(U)$ with $\gamma = 1 - n/p$

<u>Example:</u> let $n = 3$ and $u \in W^{2,2}$. Then $u, Du \in W^{1,2}$
$p = 2 < 3 = n \Rightarrow p^* = \frac{3 \cdot 2}{3-2} = 6$
(GNS)
$\Longrightarrow u, Du \in L^6 \Rightarrow u \in W^{1,6}$ and $6 > 3$ so $\gamma = 1 - n/p = \frac{1}{2}$ and $u \in C^{0,1/2}$.

<u>Chapter 4: Second order BVPs.</u>

In this entire chapter, let U be a nice domain and $\partial U \in C^1$. For $u \in C^2(\bar{U})$,

$$Lu = -\sum_{i,j=1}^n (a_{ij}(x) u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c$$

Here a^{ij}, b^i, c are given f^{n}s on U. We assume, at least $\in L^\infty(U)$. Wlog $a^{ij} = a^{ji}$.
This form is called <u>divergence form</u> $(= \nabla \cdot (A \nabla u))$. If $a^{ij} \in C^1(\bar{U})$ then one can rewrite L in <u>non-divergence form</u> $La = -\sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^n \tilde{b}^i(x) u_j + c \cdot u$

Form (1) $\Longrightarrow$ Hilbert space methods
" (2) $\longrightarrow$ max principles, Dirichlet energies
$\to$ Elliptic PDEs.

<u>Def:</u> Say L is <u>elliptic</u> if $\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j > 0$ $\forall x \in U$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Say that L is <u>uniformly elliptic</u> if $\sum a^{ij}(x) \xi_i \xi_j \ge \theta |\xi|^2$ $\forall x \in U, \xi \in \mathbb{R}^n$ and some $\theta > 0$.
(independent of x, ξ)

<u>Weak formulation + Lax. Milgram:</u>

We consider the BVP $\begin{cases} Lu = f & \text{in } U \\ u|_{\partial U} = 0. \end{cases}$ (1)

with $f \in L^2(U)$, $a^{ij}, b^j, c \in L^\infty(U)$.
Suppose $u \in C^2(\bar{U})$ solves (1) pointwise a.e. Take any $v \in C^2(\bar{U})$ with $v|_{\partial U} = 0$. We get

$$\int_U f v \, dx = \int_U v \left[(-a^{ij} u_{x_i})_{x_j} + b^i u_{x_i} + c \cdot u \right] dx$$

$$= \underbrace{- \int_{\partial U} v a^{ij} u_{x_i} n_j \, dS}_{0 \text{ as } v|_{\partial U} = 0} + \int_U a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + c u v \, dx$$

(2) $\Rightarrow \left[\int_U v f \, dx = B[u,v] = \int_U a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + c u v \, dx \right.$

So if $u \in C^2(\bar{U})$ solves (1), then (2) holds.
Conversely, if $u \in C^2(\bar{U})$, $u|_{\partial U} = 0$ and (2) holds, then by undoing the integration by parts we get $\int_U [f - Lu] v \, dx = 0$ $\forall v \in C_c^\infty(U)$

$\Rightarrow Lu = f$ pointwise a.e.
<u>Conclusion:</u> if $u \in C^2(\bar{U})$, $u|_{\partial U} = 0$ then u solves (1) $\Longleftrightarrow$ u solves (2).
<u>Key</u>: (2) makes sense for $v \in H_0^1(U)$ and $u \in H^1$. To encode the BC's $\Rightarrow u \in H_0^1(U)$.
$H^k = W^{k,2}$.

<u>Defn:</u> We say $w \in H_0^1(U)$ is a <u>weak solution</u> of the BVP $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$
for given $f \in L^2(U)$ if $B[u,v] = (f,v)_{L^2(U)}$ $\forall v \in H_0^1(U)$.

<u>Theorem</u>: (Lax–Milgram 1954) Let H be a real Hilbert space with inner product (IP) $(\cdot,\cdot)$. Suppose $B: H \times H \to \mathbb{R}$ is a bilinear map s.t. $\exists$ constants $\alpha, \beta > 0$ s.t.
(i) $|B[u,v]| \le \alpha \cdot \|u\| \cdot \|v\|$ $\forall u,v \in H$. (boundedness)
(ii) $\beta \|u\| \le B[u,u]$ $\forall u \in H$ (coercivity).

Then if $f: H \to \mathbb{R}$ is a bdded linear functional ($f \in H^*$). Then $\exists!$ $u \in H$ s.t. $B[u,v] = \langle f, v \rangle$ $\forall v \in H$.

<u>Example:</u> Recall $H^k = W^{k,2}$ are Hilbert spaces.
Consider $\begin{cases} Lu = -\Delta u + cu, \; c \ge 0, & \text{in } U. \\ u = f \in L^2 & \text{on } \partial U. \end{cases}$

$$B[u,v] = \int_U (\nabla u \cdot \nabla v + c \cdot u \cdot v) \, dx$$

(i) $|B[u,v]| \le (1+c) \cdot \|u\|_{H^1} \cdot \|v\|_{H^1}$
(ii) $|B[u,u]| = \left| \int_U |\nabla u|^2 + c u^2 \, dx \right| = \|\nabla u\|^2_{L^2(U)} + c\|u\|^2_{L^2(U)}$
$\underset{c \ge 0}{\ge} \|\nabla u\|^2_{L^2(U)} \ge \tilde{c} \|u\|^2_{H^1(U)}$.
(Lax Milgram with Hilbert space $= H_0^1$)

<u>Corollary</u>: ("Stability of Lu") Suppose Lax–Milgram (LM). Let u_i be the unique sol^n to $B[u_i, v] = \langle f_i, v \rangle$ $\forall v \in H$.
Then $\|u_1 - u_2\| \le \frac{1}{\beta} \|f_1 - f_2\|_{H^*}$

<u>Proof:</u> Since $B[u_i, v] = \langle f_i, v \rangle$ $\forall v \in H$ and $i = 1, 2$
$\Rightarrow B[u_1 - u_2, v] = \langle f_1 - f_2, v \rangle$ $\forall v \in H$
choose $v = u_1 - u_2$. Then $\beta \|v\|^2 = \beta \|u_1 - u_2\|^2$
$\le B[v,v] = \langle f_1 - f_2, u_1 - u_2 \rangle \le \|f_1 - f_2\|_{H^*} \cdot \|v\|$
Divide through by $\|v\|$ to get conclusion

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Proof: (Lax-Milgram)

① For each fixed $u \in H$, the map $\varphi_u(v) = B[u,v]$ is a bdd linear functional on H, i.e. $\varphi_u \in H^*$. By Riesz Rep. Thm, $\exists!\, w_u \in H$ s.t. $\varphi_u(v) = (w_u, v) = B[u,v]$ $\forall v \in H$. So there is a map $u \mapsto w_u \in H$ which we denote $A: H \to H$, $u \mapsto w_u$ and we have

$B[u,v] = (Au, v)$ $\forall v \in H$.

② We first show A is a bdd linear map. If $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$, then for each $v \in H$, we have $(A(\lambda_1 u_1 + \lambda_2 u_2), v)$

$= B[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v]$

$= (\lambda_1 A u_1 + \lambda_2 A u_2, v)$ $(\forall v \in H)$

$\Rightarrow A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 A u_1 + \lambda_2 A u_2 \Rightarrow A$ linear.

Also, $\|Au\|^2 = (Au, Au) = B[u, Au] \le \alpha \|u\| \cdot \|Au\|$

$\longrightarrow \|Au\| \le \alpha \cdot \|u\|$ $\forall u \in H$

$\Rightarrow A$ is bdd.

③ Now show A injective and $A(H)$ is closed.

$$\beta \|u\|^2 \le B[u,u] = (Au, u) \le \|Au\| \cdot \|u\|$$

$$\Rightarrow \beta \|u\| \le \|Au\| \Rightarrow \|u\| \le \frac{1}{\beta} \cdot \|Au\|.$$

If $Au_1 = Au_2$, then $\|u_1 - u_2\|^2 \le 0 \Rightarrow u_1 = u_2$.

$\Rightarrow (u_j)_j$ is Cauchy in the complete space H.

$\Rightarrow u_j \to u \in H$. By the continuity of A.

$\lim_j A(u_j) = A(\lim_j u_j) \in Im(H) \Rightarrow w = A(u)$.

$\Rightarrow A(H)$ is closed in H.

④ $A(H) = H$.

Since $A(H)$ is closed, and H a Hilbert space.

$H = A(H) \oplus A(H)^\perp$.

If $A(H) \ne H$, then $\exists\, w \in A(H)^\perp$ s.t. $w \ne 0$.

But then $\beta \|w\|^2 \le B[w,w] = (Aw, w) = 0$

$\Rightarrow \|w\| = 0 \Rightarrow w = 0$ ⚡

So A is bijective and A^{-1} exists.

We define $w = Au \Leftrightarrow u = A^{-1} w$

$\|u\| \le \frac{1}{\beta} \|Au\| \Rightarrow \|A^{-1}(w)\| \le \frac{1}{\beta} \|w\|$.

$\Rightarrow A^{-1}: H \to H$ is linear & bdd.

⑤ We want to solve the following problem: given $f \in H^*$ find u s.t. $B[u,v] = \langle f, v \rangle$ $\forall v \in H$. By the Riesz, $\exists!\, w_f \in H$ s.t. $\langle f, v \rangle = (w_f, v)$ $\forall v \in H$.

Let $u = A^{-1}(w_f)$. We know this exists by ④. Then, $B[u,v] = (Au, v) = (w_f, v)$

$= \langle f, v \rangle$ $\forall v \in H$, i.e. $B[u, \cdot] = f$.

⑥ For uniqueness if both u_1 and u_2 satisfy $B[u_1, v] = \langle f, v \rangle = B[u_2, v]$ $\forall v \in H$.

$\Rightarrow B[u_1 - u_2, v] = 0$ $\forall v \in H$. Set $v = u_1 - u_2$, then $\beta \|u_1 - u_2\|^2 \le B[u_1 - u_2, u_1 - u_2] = 0$

$\Rightarrow u_1 = u_2$. □

Theorem: (4.2) (Energy estimates for $B[\,,\,]$)

Suppose $Lu = -(a^{ij} u_{x_i})_{x_j} + b^i u_{x_i} + cu$. Suppose $a^{ij} = a^{ji}$, $b^i, c \in L^\infty(U)$, suppose L is uniformly elliptic.

Then if $B[u,v] = \int_U (a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + cuv)\,dx$

then, $\exists$ constants, $\alpha, \beta > 0$ and a constant $\gamma \ge 0$ s.t.

(i) $|B[u,v]| \le \alpha \|u\|_{H_0^1(U)} \cdot \|v\|_{H_0^1(U)}$

(ii) $\beta \|u\|^2_{H_0^1(U)} \le B[u,u] + \gamma \|u\|^2_{L^2(U)}$

$\forall u, v \in H_0^1(U)$.

→ Garding's inequality.

Proof: (i) $|B[u,v]| \le \sum_{i,j} \|a^{ij}\|_{L^\infty(U)} \int_U |Du| \cdot |Dv|\,dx$

$+ \sum \|b^i\|_{L^\infty(U)} \int_U |Du| \cdot |v| + \|c\|_{L^\infty(U)} \int |u| \cdot |v|\,dx$.

Constants
$\lesssim \alpha \cdot \|u\|_{H_0^1(U)} \cdot \|v\|_{H_0^1(U)}$ for some $\alpha > 0$.

(ii) Now we use uniform ellipticity.

$$\theta \int_U |Du|^2 dx \le \int_U \sum a^{ij} u_{x_i} u_{x_j}\,dx$$

$$= B[u,u] - \int (b^i u_{x_i} \cdot u + cu^2)\,dx$$

$$\le B[u,u] + \sum \|b^i\|_{L^\infty(U)} \int_U |Du| \cdot |u|\,dx$$

$$+ \|c\|_{L^\infty(U)} \cdot \int_U u^2 dx.$$

By Young's inequality, $\left(|ab| = \sqrt{2\varepsilon}\, |a| \cdot \frac{|b|}{\sqrt{2\varepsilon}} \right)$

$$\int_U |Du| \cdot |u| \le \varepsilon \int_U |Du|^2 dx + \frac{1}{\varepsilon} \int_U |u|^2 dx.$$

Chose ε s.t. $\varepsilon \cdot \sum \|b^i\|_{L^\infty(U)} < \theta/2$.

$\Rightarrow \theta/2 \int_U |Du|^2 dx \le B[u,u] + C \|u\|^2_{L^2(U)}$.

Add to this the Poincaré inequality $\|u\|_{L^2} \le C \|Du\|_{L^2}$

$\Rightarrow \beta \|u\|^2_{H_0^1(U)} \le B[u,u] + \gamma \|u\|^2_{L^2(U)}$.

for some $\beta > 0$, $\gamma \ge 0$.

Remark: if B is a bilinear form to the operator with $b^i = c = 0$, then

$$\theta \int_U |Du|^2 dx \le B[u,u].$$

together with Poincaré $\|u\|^2_{H_0^1(U)} \le c \cdot B[u,u]$, (i.e. Garding with $\gamma = 0$).

$\Rightarrow$ apply Lax-Milgram directly.

If $\gamma > 0$, then we don't have conditions of Lax-Milgram. This motivates the following:

Theorem 4.3: Let L be as before. There is a $\gamma \ge 0$ s.t. for any $\mu \ge \gamma$ and any $f \in L^2(U)$, there exists a unique weak sol$^{\underline{n}}$ $u \in H_0^1$ to the BVP $\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ — ③

Moreover, $\exists\, C > 0$ s.t. $\|u\|_{H^1(U)} \le C \cdot \|f\|_{L^2(U)}$.

Proof: ① Take γ from Garding's inequality.

$\beta \|u\|^2_{H_0^1(U)} \le B[u,u] + \gamma \|u\|^2_{L^2(U)}$.

Let $\mu \ge \gamma$ and set $B_\mu[u,v] = B[u,v] + \mu(u,v)_{L^2}$

This is the bilinear form corresponding to $L_\mu u = Lu + \mu u$. Also can check (exercise!) that B_μ satisfies the conditions of Lax-Milgram.

$$\beta \|u\|^2 \le B_\mu[u,v]$$

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Theorem (last time) Given L, U as in Garding inequality. There is a $\gamma \geq 0$ s.t. for any $\mu \geq \gamma$ and $f \in L^2(U)$ then $\exists!$ sol^n $u \in H_0^1(U)$ to the BVP

③ $-\begin{cases} Lu + \mu u = f, & U \\ u = 0, & \partial U \end{cases}$

and $\|u\|_{H^1(U)} \leq C \cdot \|f\|_{L^2(U)}$.

Proof:

① From Garding's inequality. $\beta \|u\|^2_{H^1(U)} \leq B[u,u]$
$+ \gamma \|u\|^2_{L^2(U)} \leq B_\mu[u,u]$ where

$B_\mu[u,v] = B[u,v] + \mu (u,v)_{L^2(U)}$.

Given $f \in L^2(U)$ and set $\langle f, v\rangle = (f,v)_{L^2(U)}$.
$\to$ This is a bounded linear functional on $L^2(U)$. I.e. $f \to (f, \cdot) \in (L^2(u))^*$.
$\Rightarrow$ bdd linear functional on $H_0^1(U)$.
Apply Lax-Milgram $\Rightarrow \exists!\ u \in H = H_0^1(U)$ s.t. $B_\mu[u,v] = \langle f, v\rangle = \langle f, v\rangle_{L^2(U)}$ $\forall v \in H_0^1(U)$.
Finally, $\beta \|u\|^2_{H_0^1(U)} \leq B_\mu[u,u] = (f,u)_{L^2(U)}$
$\leq \|f\|_{L^2(U)} \cdot \|u\|_{L^2(U)} \leq \|f\|_{L^2(U)} \cdot \|u\|_{H_0^1(U)}$

$\Rightarrow$ divide by $\|u\|_{H_0^1(U)}$.

Sol^n only in H_0^1, $\mu \to$ pay a price

Compactness results in PDE:

Bolzano-Weierstrass Theorem:

The closed unit ball in $\mathbb{R}^n$ is sequentially compact.

In a vector space, compactness $\Leftrightarrow$ sequential compactness. Hilbert spaces have metrics.

If H is infinite dimensional, then $B_H = \{x \in H \mid \|x\| \leq 1\}$ is not compact.

$\Rightarrow$ resolution is to weaken the topology. i.e. topology induced by $\|\cdot\|$ is too strong.

Def^n: Spse $(H, (\cdot,\cdot))$ is a Hilbert space with $(u_j) \subset H$.
We say u_j converges weakly to $u \in H$, $u_j \rightharpoonup u$ if $\lim_j (u_j, w) = (u, w)$ $\forall w \in H$.

Remark: A weak limit, if it exists is unique. Spse $u_j \rightharpoonup u$, and $u_j \rightharpoonup \tilde{u}$. Then, $(u - \tilde{u}, w)_H = \lim_j (u_j - u_j, w)_H = 0$
Holds true $\forall w \in H \Rightarrow u = \tilde{u}$.

Corollary (Banach-Alaoglu for separable Hilbert space)
Let H be a sep. Hilbert space and spse $(u_n) \subset H$ is a bdded seq. (i.e. $\|u_n\| \leq K$). Then (u_n) has a weakly convergent subsequence. I.e., the closed unit ball in H is weakly sequentially compact.

Theorem (Banach-Alaoglu) Let X be a Banach space and consider, the closed unit ball in X^* is compact in the weak-* topology on X^*.

Lemma: (Poincaré Revisited). Suppose $u \in H^1(\mathbb{R}^n)$ and let $Q = (\xi_1, \xi_1 + L) \times \dots \times (\xi_n, \xi_n + L)$ be a cube of side lengths L. Then (i)

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{|Q|}\left(\int_Q u\,dx\right)^2 + \frac{nL^2}{2}\|Du\|^2_{L^2(U)}.$$

(ii) $$\|u - \bar{u}\|_{L^2(Q)} \leq \frac{nL^2}{2}\|Du\|^2_{L^2(U)}, \quad \bar{u} = \frac{1}{|Q|}\int_Q u(x)\,dx.$$

if $\bar{u} = 0$, get prev. Poincaré ineq.

Proof (i) Since ∂Q is Lipschitz, we apply the approx. theorem i.e. $C^\infty(\bar{Q})$ are dense in $H^1(Q)$. Consider $u \in C^\infty(\bar{Q})$. For $x, y \in Q$, we use the FTC to write

$$u(x) - u(y) = \int_{y_1}^{x_1} \frac{d}{dt} u(t, x_2, \dots, x_n)\,dt$$
$$+ \int_{y_2}^{x_2} \frac{d}{dt}\big(u(y_1, t, x_3, \dots, x_n)\big)\,dt + \dots + \int_{y_n}^{x_n} \frac{d}{dt} u(y_1, \dots, y_{n-1}, t)\,dt$$

Cross terms Cancel

Square this identity

$$(u(x) - u(y))^2 = u(x)^2 + u(y)^2 - 2u(x)u(y)$$
C-S
$$\leq n\left(\int_{y_1}^{x_1} \frac{du}{dt}(t, x_2, \dots, x_n)\,dt\right)^2$$
$$+ \dots + n\left(\int_{y_n}^{x_n} \frac{du}{dt}(y_1, \dots, y_{n-1}, t)\,dt\right)^2$$

Integrate $x, y \in Q$:

$$LHS = \int_Q dx \int dy = 2|Q| \cdot \|u\|^2_{L^2(U)} - 2\left(\int_Q u(x)\,dx\right)^2$$ Fubini

$$I_1 = \left(\int_{y_1}^{x_1} \frac{du}{dt}(t, x_2, \dots, x_n)\,dt\right)^2 \leq (x_1 - y_1) \cdot \int_{y_1}^{x_1}\left(\frac{du}{dt}\right)^2 dt$$ C-S
$$\leq L \cdot \int_{\xi_1}^{\xi_1 + L}\left(\frac{du}{dt}(t, x_2, \dots, x_n)\right)^2 dt$$

$$\longrightarrow \int_Q dx \int_Q dy\, I_1 \leq L \cdot L \cdot |Q| \cdot \|D_1 u\|^2_{L^2(Q)}.$$

All together:
$$2 \cdot |Q| \cdot \|u\|^2_{L^2(Q)} - 2\left(\int_Q u(x)\,dx\right)^2 \leq L^2 n |Q| \cdot \|Du\|^2_{L^2(Q)}$$
(Rearrange and done)

(ii) Consider $\eta \in C_c^\infty$ s.t. $\eta = 1$ on Q. then
$$\int_Q (u - \bar{u}\eta)\,dx = 0 \Rightarrow \text{result into (i)}.$$

$$\boxed{1 \leq p < n, \quad W^{1,p} \hookrightarrow L^{p^*}, \quad W^{1,p} \subset\subset L^q, \text{ where } 1 \leq q < p^*}$$

Theorem (Rellich-Kondrachov Thm). Spse $U \subset \mathbb{R}^n$ is open and bdded with $\partial U \in C^1$. Let (u_n) be a bdded sequence in $H^1(U)$ (i.e. $\|u_n\| \leq K$). Then $\exists u \in H^1(U)$ and a subsequence (u_{n_j}) s.t. $u_{n_j} \rightharpoonup u$ in $H^1(U)$, $u_{n_j} \to u$ $L^2(U)$.

Proof: By extension theorem $\exists$ extension $\bar{u}_n \in H^1(\mathbb{R})$, supp$(\bar{u}_n) \subset\subset Q$, for some cube $Q \supset U$ and $E: H^1(U) \to H_0^1(Q)$ satisfies $\|\bar{u}_n\|_{H^1(Q)} \leq C \cdot \|u_n\|_{H^1(U)} \leq C \cdot K$.
Since $H_0^1(Q)$ is a separable Hilbert space (sheet 3). By Banach-Alaoglu, $\exists u \in H_0^1(Q)$ s.t. $\bar{u}_{n_j} \rightharpoonup u$ in $H_0^1(Q)$ and
$\|u\|_{H^1(Q)} \leq C$

Claim: $w_j = \bar{u}_{n_j} \to u$ in $L^2(Q)$.

Pf: Fix $\delta > 0$. Divide Q into $k(\delta)$ subcubes $\{Q_a\}_{a=1}^k$ of side-length $0 < \delta < L$, intersecting only on their faces

$$\|w_j - u\|^2_{L^2(Q)} = \sum_{a=1}^k \|w_j - u\|^2_{L^2(Q_a)}$$
Poincaré
$$\leq \sum_{a=1}^k \left(\frac{1}{|Q_a|}\left(\int_{Q_a}(w_j - u)\,dx\right)^2\right) + \frac{n\delta^2}{2}\|Dw_j - Du\|^2_{L^2(Q)}.$$

Let $\varepsilon > 0$, Since $w_j, u \in H_0^1(Q)$, we have $\|Dw_j - Du\|^2_{L^2(Q)} \leq C$. Take $\delta > 0$ small s.t.

$\frac{n^2\delta^2}{2}\|Dw_j - Du\|^2_{L^2(Q)} < \varepsilon/2$. Fix such δ, this fixes $k(\delta)$. Note $f \mapsto \int_{Q_a} f(x)\,dx$ is a bdded linear functional on $H_0^1(Q)$ $\Rightarrow$ by $w_j \rightharpoonup u$ in $H_0^1(Q)$ so we have $\int_{Q_a}(w_j - u)\,dx \to 0$ for all a. Since $k(\delta)$ is finite & fixed we choose j large enough s.t.
$$\sum_{a=1}^k \frac{1}{|Q_a|}\left(\int_{Q_a}(w_j - u)\,dx\right)^2 < \varepsilon/2$$

$$\Rightarrow \|w_j - u\|^2_{L^2(Q)} < \varepsilon. \qquad \square$$

ANALYSIS OF PDE
LECTURE 15

Fredholm Alternative & spectra of elliptic PDE

<u>Def</u>$^{\underline{n}}$: Let H be a Hilbert space, $K : H \to H$ a bounded linear operator. The adjoint of K, $K^+ : H \to H$ is the unique operator s.t.
$(x, K^+ y) = (Kx, y)$ $\forall x, y \in H$.
K is called compact if for each bdd sequence $(u_j) \subset H$, $\exists$ a subsequence $(u_{j_k}) \subset H$ s.t. $(K(u_{j_k}))_k$ converges <u>strongly</u> in H.

<u>Key example:</u> Let $K : L^2(U) \to H^1(U)$ be a bded linear operator. Since $H^1 \hookrightarrow L^2$, can think of $K : L^2(U) \to L^2(U)$. <u>Claim:</u> $K : L^2 \to L^2$ is compact.

<u>Pf:</u> If $(u_j)_j \subset L^2(U)$ a bded seq. Then $\|K(u_j)\|_{H^1(U)} \leq \|K\| \cdot \|u_j\|_{L^2(U)} \leq C \cdot K$.
$\Rightarrow$ By Rellich–Kondrachov, $\exists$ a subsequence $(u_{j_k}) \subset H^1(U)$ s.t.
$u_{j_k} \longrightarrow u$ (strongly) in $L^2(U)$, i.e. $K(u_{j_k})$ converges strongly in $L^2(U)$.

<u>Idea:</u> $\Delta u = f$, is a map $H^1(U) \to L^2(U)$
$u \mapsto f$.
Finding a soln is the inverse map $K : L^2(U) \to H^1(U)$, $f \mapsto u$, is compact by key example above.

<u>Theorem 4.6</u> (Fredholm alternative for compact operators).
Let H be Hilbert, $K : H \to H$ be a compact linear operator.
(i) $\ker(I-K)$ is finite dimensional.
(ii) $\operatorname{Im}(I-K)$ is closed
(iii) $\operatorname{Im}(I-K) = \ker(I-K^+)^\perp$
(iv) $\ker(I-K) = \{0\} \Leftrightarrow \operatorname{Im}(I-K) = H$.
(v) $\dim(\ker(I-K)) = \dim(\ker(I-K^+))$.
<u>Pf:</u> $\to$ Appendix D.5 of Evans.
(iii), (iv) are referred to the Fredholm alternative.

<u>Applied to linear algebra:</u> $Ax = b$.
either (a) $\ker A = \{0\}$, so A^{-1} exists and so the <u>inhomogeneous</u> problem $Ax = b$ has a unique sol$^{\underline{n}}$
or (b) $\ker(A) \neq \{0\}$, i.e. the <u>hom</u> problem $Ax = 0$ admits non-trivial solns. Moreover, $\operatorname{Im}(A) = (\ker A^T)^\perp$, so the inhomogeneous problem $Ax = b$ has a solution if $b \in (\ker A^T)^\perp$, i.e. $y^T b = \langle y, b \rangle = 0$ $\forall y \in \ker A^T$, i.e. $A^T y = 0$.

Restate (iii) (iv) from Fredholm: either
(I) for each $f \in H$, $(I-K)u = f$ has a unique solution.
(II) the homogeneous eqn. $(I-K)u = 0$ has non-trivial solutions and in this case, the space of homogeneous operators is finite dim and $(I-K)u = f$ has a soln $\Longleftrightarrow f \in \ker(I-K^+)^\perp$

<u>Def</u>$^{\underline{n}}$: H is a real Hilbert space, $A : H \to H$ bded linear operator, the <u>resolvent (set)</u> of A is $\rho(A) := \{\lambda \in \mathbb{R} \mid (A - \lambda I) \text{ is invertible}\}$. Can show $\rho(A)$ is open in H. The <u>real spectrum</u> of A, is $\sigma(A) = \mathbb{R} \setminus \rho(A)$ so closed.
We say $\eta \in \sigma(A)$, belongs to the <u>point spectrum</u> of A, $\sigma_p(A)$, if $\ker(A - \eta I) \neq \{0\}$, i.e. $\exists w \neq 0$ s.t. $Aw = \eta w$ and call w an <u>eigenvector</u>

Say A is <u>self-adjoint</u> if $A = A^+$, i.e. $(Ax, y) = (x, Ay)$ $\forall x, y \in H$.

<u>Theorem:</u> (Spectrum of compact operator).
Assume H is a separable infinite-dim Hilbert space with $K : H \to H$ compact. Then,
(i) $0 \in \sigma(K)$
(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$
(iii) $\sigma(K) \setminus \{0\}$ is at most countable. $\{\lambda_i\}_{i \in \mathbb{N}}$ and if it is infinite, then $\lambda_i \to 0$
(iv) If K is self-adjoint, then $\exists$ a countable orthonormal basis for H consisting of eigenvectors of K.

<u>Applications to elliptic BVP:</u>

$$Lu = -\sum (a^{ij}(x) u_{x_i})_{x_j} + \sum b^i(x) u_{x_i} + cu.$$

uniformly elliptic on $U \subset \mathbb{R}^n$. The bilinear form assoc. to L is $B[u, v] := \int_U a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + cuv\,dx$

<u>Def</u>$^{\underline{n}}$: We define the <u>formal adjoint of L</u>:

$$L^+ v = -\sum (a^{ij} v_{x_i})_{x_j} - \sum b^i v_{x_i} + (c - \sum b^i_{x_i}) v$$

The <u>adjoint bilinear form:</u>

$B^+ : H_0^1(U) \to H_0^1(U)$ is given by
$B^+[v, u] = B[u, v]$.

We say $v \in H_0^1(U)$ is a weak sol$^{\underline{n}}$ of the <u>adjoint problem</u> $\begin{cases} L^+ v = f \text{ in } U \\ v = 0 \text{ on } \partial U \end{cases}$ if it satisfies $B^+[w, v] = (f, w)_{L^2(U)}$ $\forall w \in H_0^1(U)$.

<u>Note:</u> if $b^i \in C^1(\bar{U})$, then B^+ is the same as the bilinear form defined by B^+.

<u>Theorem 4.8:</u> (Fredholm alternative for elliptic BVP).
Consider ① $- \begin{cases} Lu = f \text{ in } U \\ u = 0 \text{ in } \partial U \end{cases}$, then <u>either</u>
(a) for each $f \in L^2(U)$, the (inhomog.) problem ① admits a unique weak sol$^{\underline{n}}$ $u \in H_0^1(U)$, OR
(b) $\exists$ a non-trivial weak sol$^{\underline{n}}$ $u \in H_0^1(U)$ to the hom. problem (i.e. $f = 0$ in ① and $\dim(N) = \dim(N^+) < \infty$ with
$N = \{\text{weak solns to the BVP}\} \subset H_0^1(U)$.
$N^+ = \{\text{weak solns to homog. adjoint BVP}\} \subset H_0^1$.
Finally, ① has a weak soln $\Leftrightarrow$ $\langle f, v \rangle_{L^2(U)} = 0$ $\forall v \in N^+$.

<u>Proof:</u> By Thm 4.3 $\exists$ $\gamma \geq 0$ s.t. for any $f \in L^2(U)$ $\exists!$ weak sol$^{\underline{n}}$ $u \in H_0^1(U)$ to
$\gamma = 0$ trivial (forget (γ) by Lax-M.
$\begin{cases} L_\gamma u = f \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases}$ where $L_\gamma u = Lu + \gamma u$.
i.e. $B_\gamma[u, v] = B[u, v] + \gamma(u, v)_{L^2} = (f, v)$ $\forall v \in H_0^1(U)$.
and $\|u\|_{H^1} \leq C \cdot \|f\|_{L^2}$
Write $L_\gamma^{-1}(f) := u$. Check this is linear, inhomogeneity $\to$ sol$^{\underline{n}}$, then $\|L_\gamma^{-1}(f)\|_{H^1} \leq C \|f\|_{L^2}$.
$\Rightarrow L_\gamma^{-1} : L^2 \to H_0^1$ is bded,
$\Rightarrow L_\gamma^{-1} : L^2 \to L^2$ is compact.
<u>Observe:</u> if $g \in L^2$, then $L_\gamma^{-1}(g) = w \Leftrightarrow B_\gamma[w, v] = (g, v)$ $\forall v \in H_0^1$.

Now, suppose $u \in H_0^1$ is a weak sol$^{\underline{n}}$ to ①
i.e. $B[u, v] = (f, v)_{L^2}$ $\forall v \in H_0^1$.
$\Rightarrow B_\gamma[u, v] = (f + \gamma u, v)_{L^2}$ $\forall v \in H_0^1$.
Then u solves ① weakly iff $u = L_\gamma^{-1}(f + \gamma u)$
$= L_\gamma^{-1}(f) + \gamma L_\gamma^{-1}(u) \Leftrightarrow u - Ku = h$, where $K = \gamma L_\gamma^{-1}$, $h = L_\gamma^{-1}(f)$.

Analysis of PDE
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Proof: for any $f \in L^2(U)$, $\exists!$ weak sol^n $u \in H_0^1$ to $\begin{cases} L_\gamma u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$. Write $L_\gamma^{-1}(f) = u$.
Recall $L_\gamma^{-1} : L^2 \to H_0^1$.
See u solves ① weakly $\Leftrightarrow (I-K)u = h$, $K = \gamma L_\gamma^{-1}$, $h = L_\gamma^{-1}(f)$.

Observe $K : L^2 \to L^2$ is also compact. Thereby Fredholm th^m for compact operators either (I) for all $h \in L^2$, $u - Ku = h$ admits a sol^n $u \in L^2$, or (II) $\exists\, 0 \neq u \in L^2$ s.t. $u - Ku = 0$.

Suppose (I) holds. Setting $h = L_\gamma^{-1}(f)$ we have $\exists\, u \in L^2(U)$ s.t. $u = \gamma L_\gamma^{-1}(f) + L_\gamma^{-1}(f)$.
Since $L_\gamma^{-1} : L^2 \to H_0^1$, we get $u \in H_0^1$ and by the above see that u is a weak sol^n of ① $= \dots$

Suppose (II): so $\exists\, u \neq 0 \in L^2$ s.t. $u = Ku$
By def^n of L_γ^{-1}, $L_\gamma^{-1}(u) \Rightarrow u \in H_0^1$. $B[u, v] + \gamma (u, v)_{L^2} = (\gamma u, v)_{L^2}$ $\forall v \in H_0^1$
$\Rightarrow B[u, v] = 0$ $\forall v \in H_0^1$, i.e. u is a weak sol^n to hom. BVP ($u \in N$)
Also by Fredholm, $\dim N = \dim(\mathrm{Ker}(I-K)) = \dim(I-K^+) = \dim N^+ < \infty$.

Claim: let $v \in L^2$, then $(I-K^+)v = 0 \Leftrightarrow B^+[v, w] = 0$ $\forall w \in H_0^1$

Pf: $(I-K^+)v = 0 \Leftrightarrow (v, w)_{L^2} = (v, Kw)_{L^2}$ $\forall w \in L^2$
$\Leftrightarrow (v, w)_{L^2} = (v, \gamma L_\gamma^{-1}(w))_{L^2}$ $\forall w \in L^2(U)$.
But a weak sol^n to $\begin{cases} L_\gamma \tilde{w} = f & \text{in } U \\ \tilde{w} = 0 & \text{on } \partial U \end{cases}$
obeys $B[\tilde{w}, \varphi] + \gamma(\tilde{w}, \varphi) = (f, \varphi)$ $\forall \varphi \in H_0^1$.
So if we take $f = w$, then we have $\tilde{w} = L_\gamma^{-1}(w)$.
$\Rightarrow B[L_\gamma^{-1}(w), v] + \gamma(L_\gamma^{-1}(w), v) = (w, v)_{L^2}$

Inserting this into ⊛ $(I-K^+)v = 0$
$\Leftrightarrow B[L_\gamma^{-1}(w), v] + \gamma(L_\gamma^{-1}(w), v)_{L^2} = (v, \gamma L_\gamma^{-1}(w))_{L^2}$ $\forall v \in L^2$
$\Leftrightarrow B[L_\gamma^{-1}(w), v] = 0$ $\forall w \in L^2$

$\Leftrightarrow B^+[v, L_\gamma^{-1}(w)] = 0$ $\forall w \in L^2$.
To finish, we need $B^+[v, \varphi] = 0$ $\forall \varphi \in X$, X dense in H_0^1.
Ex. Sheet 3: $\mathrm{Im}(L_\gamma^{-1})$ is dense in H_0^1 $\Rightarrow$ by cont^n L_γ^{-1} and so we have shown.
$(I - K^+)v = 0 \Leftrightarrow B^+[v, w] = 0$ $\forall w \in H_0^1$

RTP that ① has a weak sol^n $\Leftrightarrow$ $(f, v)_{L^2} = 0$ $\forall v \in N^+$.
① has a sol^n $\Leftrightarrow (I-K)u = L_\gamma^{-1}(f)$
$\Leftrightarrow L_\gamma^{-1}(f) \in \mathrm{Im}(I-K) \underset{\text{Fredholm}}{=} \mathrm{Ker}(I-K^+)^\perp$
$\Leftrightarrow (v, L_\gamma^{-1}(f))_{L^2} = 0$
$\forall v \in \mathrm{Ker}(I-K^+)$. But $\forall v \in \mathrm{Ker}(I-K^+)$,
$0 = (v, L_\gamma^{-1}(f))_{L^2} = (v, \frac{1}{\gamma} K(f))_{L^2} = \frac{1}{\gamma}(K^+ v, f)_{L^2}$
$= \frac{1}{\gamma}(v, f)_{L^2}$

hence $(v, f)_{L^2} = 0$ $\forall v \in \mathrm{Ker}(I-K^+)$.

Remark: given L, see for γ large, L_γ is bounded invertible linear map.
Typically, $L_\gamma^{-1} = (L + \gamma I)^{-1}$ is called the resolvent of L. The fact $L_\gamma^{-1} : L^2 \to L^2$ is compact is expressed by saying L has compact resolvent.

Theorem 4.9: Under the same assumption of Th^m 4.8
(i) $\exists$ an at most countable set $\Sigma \subset \mathbb{R}^n$ s.t. the BVP ② $\sim \begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$
has a weak sol^n $\forall f \in L^2$ iff $\lambda \notin \Sigma$.
(ii) if Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$ and (after reordering) $\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_k < \dots$
with $\lambda_k \to \infty$ as $k \to \infty$.
(iii) To each $\lambda \in \Sigma$ there is a finite-dim space
$$\mathcal{E}(\lambda) = \left\{ u \in H_0^1 \;\middle|\; u \text{ is a weak sol}^n \text{ to } \begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \right\}$$
We say $\lambda \in \Sigma$ is an eigen-value of L and $u \in \mathcal{E}(\lambda)$ are corresponding eigenfunctions.

e.g. $L = -\Delta + V(x)$, $U \to \mathbb{R}^n$.
$(|x|^2)$

Pf: Pick $\gamma > 0$ as in Th^m 4.8. Pick $\mu \geq \gamma$. Then $L_\mu = L + \mu I$ is invertible and $L_\mu^{-1} : L^2 \to L^2$ is compact. If $\lambda \leq -\gamma$ ($\mu = -\lambda \geq \gamma$) then the problem
$\begin{cases} Lu - \lambda u = f \\ u = 0 \end{cases}$ admits a unique weak solution $\forall f \in L^2$ (Th^m 4.3).

$(L - \lambda I)$ is invertible $\quad -\gamma \qquad \mathbb{R} \Rightarrow \Sigma \subset (-\gamma, \infty)$.

If $\lambda > -\gamma$ then solving eqn 2 $\Leftrightarrow$ solving
② $\sim \begin{cases} (L - \lambda I)u = f & \text{in } U \text{ so} \\ u = 0 & \text{on } \partial U. \end{cases}$
apply Fredholm (Thm 4.8) to $(L - \lambda I)$. So we see ② has a unique weak solution $\forall f \in L^2$
$\Leftrightarrow$ $u = 0$ is the unique solution to
$\begin{cases} (L - \lambda I)u = 0, & \text{in } U \\ u = 0, & \text{in } \partial U \end{cases}$ (i.e. case (b) of Th^m 4.8 does not occur).
$\Leftrightarrow$ $u = 0$ is the only solution to $\begin{cases} Lu + \gamma u = (\gamma + \lambda)u \\ u = 0 \; \partial U \end{cases}$
$\Leftrightarrow$ $u = 0$ is the only sol^n to
$u = L_\gamma^{-1}((\lambda + \gamma)u)$
$= \left(\frac{\gamma + \lambda}{\gamma}\right) K(u)$.

$\Leftrightarrow$ $u = 0$ is the only sol^n to $K(u) = \frac{\gamma}{\gamma + \lambda} u$.

$\Leftrightarrow$ $\frac{\gamma}{\gamma + \lambda}$ is not an eigenvalue of K.

Then $\lambda \in \Sigma \Leftrightarrow \mu = \frac{\gamma}{\gamma + \lambda}$ is an eigenvalue of K.
By Thm 4.7, the set of e'vals of K consists of finite set or else values of a seq $\to 0$. If the set $\{\mu_k\}$ is infinite then $\mu_k \to 0 \Rightarrow \lambda_k = \frac{\gamma}{\mu_k} - \gamma \to \infty$
$\mathcal{E}(\lambda)$ is finite dim follows from Fredholm alternative (dim $N < \infty$).

Remark: if $\lambda \notin \Sigma$ then $\exists\, c > 0$ s.t. $\|u\|_{L^2} \leq c \cdot \|f\|_{L^2}$. This constant blows up as $\lambda \to$ e'value in Σ.

Self-adjoint, positive operators

Def^n: The operator L is said to be formally self-adjoint if $L = L^+$

Ex: this is equivalent to $b^i \equiv 0 \Rightarrow B[v, u] = B[u, v]$.

Def^n: L is positive if $\exists\, \beta > 0$ s.t. $\beta \|u\|_{H^1}^2 \leq B[u, u]$ $\forall u \in H_0^1$, i.e. coercive.

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Theorem 4.13: (Eigenvalues of symmetric Elliptic operators).
Let L be a uniformly elliptic, formally self-adjoint positive operator on some domain U.
Then we can represent the eigenvalues of L a sequence
$$0 < \lambda_1 \le \lambda_2 \le \dots$$
where each λ appears according to its multiplicity, $\dim(E(\lambda))$, and $\exists$ an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ for $L^2(U)$ of eigenfunctions,
$$\begin{cases} Lw_k = \lambda_k w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}$$
$w_k \in H_0^1$.

Proof: By positivity, Lax-Milgram $\Rightarrow$ L is invertible, $L^{-1}: L^2(U) \to H_0^1(U) \hookrightarrow L^2(U)$.
Denote $S := L^{-1} : L^2(U) \to L^2(U)$. S is compact (L.15).
Claim: S is self-adjoint.
pf: Pick $f, g \in L^2(U)$, then $S(f) = u$ means that $u \in H_0^1(U)$ is the unique weak solution to $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ & similarly for $S(g) = v$.

[i.e. $B[u,w] = (f,w) \ \forall w \in H_0^1$,
$B[v,\varphi] = (g,\varphi) \ \forall \varphi \in H_0^1$]

By defn of weak solution
$$(Sf, g)_{L^2} = (u, g)_{L^2} \overset{\varphi = u}{=} B[v,u]$$
& $(f, Sg)_{L^2} = (f, v)_{L^2} = B[u,v]$.

But L was self-adjoint, so $B[u,v] = B[v,u]$
i.e. $(f, Sg)_{L^2} = (Sg, f)_{L^2} \ \forall f, g \in L^2$ $\square$

Now, by thm 4.7, for compact self-adjoint operators, $\exists (\mu_k)_k \subset \mathbb{R}$ s.t. $\mu_k \to 0$ & $\exists w_k \in L^2(U)$ s.t. $(w_k)_k$ orthonormal basis for $L^2(U)$ with
$$Sw_k = \mu_k w_k \iff L^{-1} w_k = \mu_k w_k \in H_0^1$$
$$\iff Lw_k = \lambda_k w_k, \ \lambda_k = \frac{1}{\mu_k}$$
Positivity of λ_k follows from positivity of L (& so S). $\square$

4.5 Elliptic Regularity

In this section suppose $U \subset \mathbb{R}^n$ open & bounded & $V \subset\subset U$.
Aim: improve regularity of weak solutions $u \in H_0^1(U)$ to $u \in C^2(\bar{U})$ to $Lu = f$.
Motivating example: $u \in C_c^\infty(\mathbb{R}^n)$ with $-\Delta u = f$
Then
$$\int_{\mathbb{R}^n} f^2 dx = \int_{\mathbb{R}^n} (\Delta u)^2 dx$$
$$= \sum_{i,j} \int_{\mathbb{R}^n} (D_i D_i u)(D_j D_j u)\, dx \overset{\text{integrate by parts twice}}{=} \sum_{i,j} \int_{\mathbb{R}^n} (D_i D_j u)^2 dx$$
$$= \|D^2 u\|^2_{L^2(\mathbb{R}^n)} \Rightarrow \|D^2 u\|_{L^2(\mathbb{R}^n)} \le \|\Delta u\|_{L^2(\mathbb{R}^n)}.$$
So all 2nd derivatives controlled in $L^2(U)$ by Δu. An issue, if $u \in H^1$ then, can't make sense of $D^2 u$ (weakly).
Definition: For $0 < |h| < \operatorname{dist}(V, \partial U)$, define the difference quotient:
$$\Delta_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}, \ i = 1, \dots, n,$$
$\forall x \in V$ & write $\Delta^h u = (\Delta_1^h u, \dots, \Delta_n^h u)$.
Remark: Suppose $u \in L^2(U)$. Then $\Delta^h u \in L^2(V)$ & $D(\Delta^h u) = \Delta^h(Du)$ i.e. if $u \in H^1(U)$ $\Rightarrow \Delta^h u \in H^1(V)$.
Lemma 4.2: Suppose $u \in L^2(U)$. Then $u \in H^1(V)$ $\iff$ $\forall h$ with $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$, have $\|\Delta^h u\|_{L^2(V)} \le C$ for some $C > 0$.
Moreover, $\exists \tilde{C} > 0$ s.t.
$$\tilde{C} \cdot \|Du\|_{L^2(V)} \le \|\Delta^h u\|_{L^2(V)} \le \tilde{\tilde{C}} \|Du\|_{L^2(V)}$$
($\Delta^h u$ is equivalent to Du in V, $\|Du\|_{L^2(V)} \simeq \|\Delta^h u\|_{L^2(V)}$).
Proof: Ex. Sheet 3.

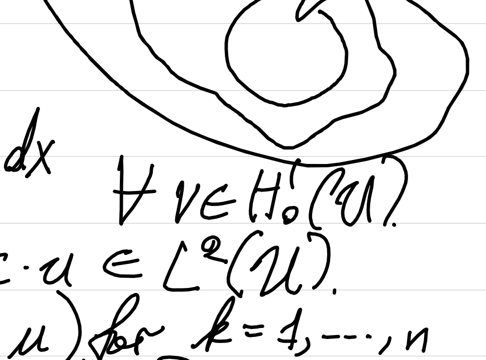
Thm 4.11: (Interior regularity)

Suppose L is uniformly elliptic on U & assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, $f \in L^2(U)$.
Suppose $u \in H^1(V)$, satisfies
(3) $B[u,v] = (f,v)_{L^2} \ \forall v \in H_0^1(U)$.

then $u \in H^2_{loc}(U)$ & for each $V \subset\subset U$ have
$$\|u\|_{H^2(V)} \le C \cdot (\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$
with $C = C(V, U, a^{ij}, b^i, c, n)$ but <u>not</u> f or u.

Remarks:
- gain 2 weak derivatives of u → very good!
- also useful to write the inequality as
$$\|u\|_{H^2(V)} \le C \cdot (\|Lu\|_{L^2(U)} + \|u\|_{L^2(U)})$$
(cf $\|D^2 u\|_{L^2} \le \|\Delta u\|_{L^2}$ for $L = \Delta$).

Proof: (1) Fix $V \subset\subset U$ and choose W compact s.t. $V \subset\subset W \subset\subset U$.
Take $\xi \in C_c^\infty(W)$, $\xi \ge 0$ s.t. $\xi|_V = 1$ (& $\xi|_{\partial W} = 0$)



Rewrite (3) as
$$\int_U a^{ij} D_i u D_j v\, dx = \int_U \tilde{f} v\, dx \quad \forall v \in H_0^1(U)$$
where $\tilde{f} = f - b^i D_i u - c \cdot u \in L^2(U)$.
Choose $v = -\Delta_k^{-h}(\xi^2 \Delta_k^h u)$ for $k = 1, \dots, n$ fixed & $0 < |h| < \frac{1}{2}\operatorname{dist}(W, \partial U)$.

Note $v \in H_0^1(W)$ and approximates $D^2 u$. Set
$$A := \int_U a^{ij} D_j u D_i v\, dx$$
$$B := \int_U \tilde{f} v\, dx,$$

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Proof: (Elliptic regularity cont^d).

Observe: For $\psi, \varphi \in L^2(U)$, supported in U. Then,

$$\int_U \psi(x)\left(\Delta_k^{-h}\varphi(x)\right)dx = -\int_U \left(\Delta_k^h \psi(x)\right)\varphi(x)\,dx$$

$\to$ IBP for diff. quotients.

Also, $\Delta_k^h(\psi\varphi)(x) = \dfrac{\psi(x+e_k h)\cdot\varphi(x+e_k h) - \psi(x)\varphi(x)}{h}$

$= (\tau_k^h\psi)(x)\,\Delta_k^h\varphi(x) + (\Delta_k^h\psi)(x)\cdot\varphi(x)$ where $\tau_k^h\psi(x) := \psi(x+he_k)$ is the translation operator.

② **Bound A:** $A = -\int_U a^{ij}u_{x_i}\,\Delta_k^{-h}\left(\left(\xi^2\Delta_k^h u\right)_{x_j}\right)dx$

$$= \int_U \Delta_k^h(a^{ij}u_{x_i})\left(\xi^2\Delta_k^h u\right)_{x_j}dx$$

$$= \int_U \left[(\tau_k^h a^{ij})\Delta_k^h u_{x_i} + (\Delta_k^h a^{ij})u_{x_i}\right]\cdot\left[\xi^2\Delta_k^h u_{x_j} + 2\xi\,\xi_{x_j}\Delta_k^h u\right]dx$$

$= A_1 + A_2$

$$A_1 = \int_U \xi^2(\tau_k^h a^{ij})(\Delta_k^h u_{x_i})(\Delta_k^h u_{x_j})\,dx$$

By uniform ellipticity, $\sum_{i,j=1}^n (\tau_k^h a^{ij}(x))\eta_i\eta_j \geq \theta|\eta|^2$ $\forall \eta\in\mathbb{R}^n$, $\forall x\in W$.

Apply with $\eta_i = \Delta_k^h u_{x_i}$, we get

$$A_1 \geq \theta\cdot\int_U \xi^2|\Delta_k^h(Du)|^2dx.$$

$$A_2 = \int_U \Big[(\Delta_k^h a^{ij})u_{x_i}\,\xi^2\Delta_k^h u_{x_j} + 2\xi\cdot(\Delta_k^h a^{ij})u_{x_i}\,\xi_{x_j}\cdot\Delta_k^h u + 2\xi\cdot(\tau_k^h a^{ij})\cdot(\Delta_k^h u_{x_i})\xi_{x_j}\Delta_k^h u\Big]dx$$

Since $a^{ij}\in C^1(\bar U)$, supp$(\xi)\subseteq W$, and continuous functions on $\bar W$ are bounded since $\bar W$ is compact.

$$\Rightarrow |A_2| \leq C_0\int_W \Big[\xi\cdot|Du|\cdot|\Delta_k^h(Du)| + \xi|Du|\cdot|\Delta_k^h u| + \xi\cdot|\Delta_k^h(Du)|\cdot|\Delta_k^h u|\Big]dx$$

Young's inequality
$$\leq \varepsilon\cdot\int_W \xi^2|\Delta_k^h(Du)|^2dx + \frac{C}{\varepsilon}\int_W |Du|^2 + |\Delta_k^h u|^2dx$$

Lemma 4.2
$$\leq \varepsilon\int_W \xi^2|\Delta_k^h(Du)|^2dx + \frac{C}{\varepsilon}\int_W |Du|^2dx$$

Set $\varepsilon = \theta/2$ and using $A_2 \geq -|A_2|$ we find $A = A_1 + A_2 \geq \frac{\theta}{2}\int_W \xi^2|\Delta_k^h(Du)|^2dx - C\int_W |Du|^2dx$

③ **Bound B:**

$$B = \int_W (f - b^i u_{x_i} - cu)v\,dx$$

$$|B| \leq C\cdot\int_W (|f| + |Du| + |u|)\,|\Delta_k^{-h}(\xi^2\Delta_k^h u)|\,dx$$

So $\int_W |\Delta_k^{-h}[\xi^2\Delta_k^h u]|^2dx \overset{L4.2}{\leq} C\int_W |D(\xi^2\Delta_k^h u)|^2dx$

$$\leq C\int_W |\xi|^2|D\xi|^2|\Delta_k^h u|^2dx + C\int_W \xi^2|\Delta_k^h(Du)|^2dx$$

$$\overset{L4.2}{\leq} C\cdot\int_W |Du|^2dx + C\int_W \xi^2|\Delta_k^h(Du)|^2dx$$

By Young's inequality on $|B|$:

$$|B| \leq \varepsilon\cdot\int_U \xi^2|\Delta_k^h(Du)|^2dx + \frac{C}{\varepsilon}\int_W \left(|f|^2 + u^2 + |Du|^2\right)dx$$

Set $\varepsilon = \theta/4$

④ $A = B \Rightarrow |A| = |B|$ so

$$\frac{\theta}{2}\int_U \xi^2|\Delta_k^h(Du)|^2dx - C\int_W |Du|^2dx \leq |A|$$

$$= |B| \leq \theta/4\int_U \xi^2|\Delta_k^h(Du)|^2dx + C\int_W (f^2 + u^2 + |Du|^2)\,dx$$

$$\Longrightarrow \int_U \xi^2|\Delta_k^h(Du)|^2dx \leq C\cdot\int_W (f^2 + u^2 + |Du|^2)\,dx$$

Since $\xi|_V = 1$ we get (in summary) if $u\in H^1(U)$ solves ③, then

$$\int_V |\Delta_k^h(Du)|^2dx \leq \int_W (f^2 + u^2 + |Du|^2)\,dx$$

Since C indep of h, by lemma 4.2, $Du\in H^1(V) \Longrightarrow u\in H^2_{loc}(U)$ with

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right).$$

⑤ Remove $\|Du\|_{L^2(V)}$ from

Let $\xi\in C_c^\infty(U)$, with $\xi|_W = 1$.

Set $v = \xi^2 u$ in eq^n ③ to get

$$\int_U \left(a^{ij}u_{x_i}(\xi^2 u)_{x_j} + b^i u_{x_i}\xi^2 u + c\cdot u^2\xi^2\right)dx = \int_U \xi^2 f\cdot u\,dx$$

As in the proof of Garding's inequality we can rearrange to get

$$\|Du\|^2_{L^2(W)} \leq C\left(B[u,u] + \gamma\|u\|^2_{L^2(W)}\right)$$

$\int fu \leq \|f\|_{L^2}\cdot\|u\|_{L^2}$

$$\leq C\left(\|f\|^2_{L^2(W)} + \|u\|^2_{L^2(W)}\right).$$

$$\Longrightarrow \|u\|_{H^1(W)} \leq C\cdot\left(\|f\|_{L^2(W)} + \|u\|_{L^2(W)}\right).$$

$$\Rightarrow \|u\|_{H^2(V)} \leq C\cdot\left(\|f\|_{L^2(W)} + \|u\|_{L^2(W)}\right)$$

Remarks:

① this is a local result: to have $u\in H^2(V)$ for $V\subset\subset U$ it is enough to have $f\in L^2(W)$, $V\subset\subset W\subset\subset U$.

i.e. if $f\notin L^2$ near ∂U then we don't see this in our estimates

② the eq^n ($Lu = f$) holds pointwise a.e.

$u\in H^2_{loc}(U)\Rightarrow Lu\in L^2_{loc}(U)$. So take $V\subset\subset U$, then $\forall v\in C_c^\infty(V)$, then we have from ③ $(Lu - f, v)_{L^2} = 0$, since $Lu - f\in L^2(V)$ so $Lu = f$ a.e. in V

Since $V\subset\subset U$ arbitrary $\Rightarrow Lu = f$ a.e. in U.

Theorem 4.12 (Higher order interior regularity)

If $a^{ij}, b^i, c\in C^{m+1}(U)$ and $f\in H^m(U)$, $m\in\mathbb{N}$, then $u\in H^{m+2}_{loc}(U)$ and $\forall V\subset\subset W\subset\subset U$.

$$\|u\|_{H^{m+2}(V)} \leq C\cdot\left(\|f\|_{H^m(W)} + \|u\|_{L^2(W)}\right).$$

Remarks: ① Hölder theory of elliptic reg:

$f\in C^{k,\alpha}(U)\Longrightarrow u\in C^{k+2,\alpha}(U)$.

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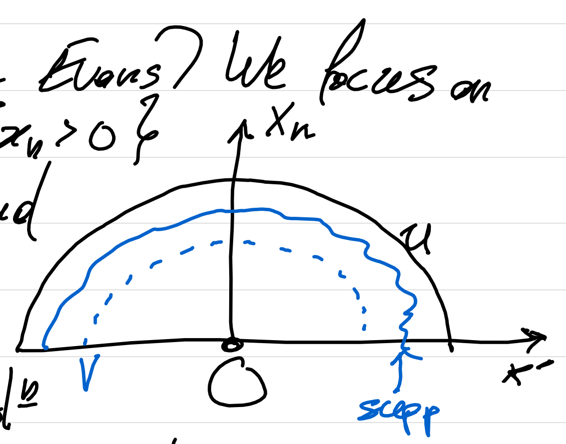
Rem ② Recall if $m > n/2$ then $H^{m+2}_{loc}(U) \hookrightarrow C^2_{loc}(U)$
$\Rightarrow$ if $f \in C^\infty(U)$ then u is also.

Theorem 4.17 (Boundary H^2 regularity)

Assume $a^{ij} \in C^1(\bar{U})$, $b^i, c \in L^\infty(U)$, $f \in L^2(U)$, $\partial C \in C^2$.
Suppose $u \in H^1_0(U)$ is a weak soln to $\begin{cases} Lu = f, & U \\ u = 0 & \partial U. \end{cases}$

then $u \in H^2(U)$ and $\|u\|_{H^2(U)} \le C \cdot (\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$.

Proof: (Sketch $\to$ See Evans) We focus on case $U = B_1(0) \cap \{x_n > 0\}$



let $V = B_{1/2}(0) \cap \{x_n > 0\}$ and choose $\xi \in C^\infty_c(B_1(0))$ with $\xi|_V = 1$, $0 \le \xi \le 1$.
Since $u \in H^1_0$ is a weak sol^n
$\Rightarrow \int_U a^{ij} u_{x_i} v_{x_j} = \int_U f v \quad \forall v \in H^1_0(U)$.

let $0 < |h| \le \frac{1}{4} \operatorname{dist}(\operatorname{supp} \xi, \partial B_1(0))$.
Consider $v = -\Delta_k^{-h}(\xi^2 \Delta_k^h u)$ for fixed $k = 1, \dots, n-1$
Claim: $v \in H^1_0(U)$.

Pf: $v(x) = -\frac{1}{h} \Delta_k^{-h}\left(\xi^2(x)\left(u(x+he_k) - u(x)\right)\right)$

for $x \in U$ $= \frac{1}{h^2}\left[\xi^2(x-he_k)(u(x) - u(x-he_k)) - \xi^2(x)(u(x+he_k) - u(x))\right]$

The translation is horizontal, $Tr(u)|_{x_n=0} = 0$.
since $u \in H^1_0(U)$
$\Rightarrow Tr(u(x \pm he_k))|_{x_n=0} = 0 \quad \forall |x| < 1-h$.
For $x_n = 0$ and $|x| \ge 1-h$, have $\xi^2(x) = 0$, $\xi(x-he_k) = 0$.

So as in the proof of Th^{m} 4.11 we deduce
$$\int_V |\Delta_k^h(Du)|^2 dx \le C \int_U (f^2 + u^2 + |Du|^2)\, dx$$
$\quad C = C(V)$.

$\Rightarrow D_k u \in H^1(V)$ for $k = 1, \dots, n-1$ with
$$\left.\begin{aligned} &\|D_k D_i u\|_{L^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}) \\ &\hookrightarrow i = 1, \dots, n \\ &\hookrightarrow k = 1, \dots, n-1. \end{aligned}\right\} \circledast$$

To control $u_{x_n x_n}$, write the PDE as
$$a^{nn} u_{x_n x_n} = F = -\sum_{i+j<2n} a^{ij} u_{x_i x_j} + \tilde{b}^i u_{x_i} + cu - f$$
holds a.e. in U. By uniform ellipticity,
$a^{nn}(x) = \sum a^{ij}(x) \eta_i \eta_j \ge \theta > 0$.
$\eta = (0, \dots, 1)$

By $\circledast$, $F \in L^2(V)$, so all together, $u_{x_n x_n} \in L^2(V)$

and $\|u\|_{H^2(V)} \le C \cdot (\|f\|_{L^2(U)} + \|u\|_{H^1(U)})$.

Again like in the proof of Garding's inequality, we can replace $\|u\|_{H^1}$ on RHS with $\|u\|_{L^2}$.
To finish, $\partial U = \bigcup_{i=1}^N V_i$ and then sum $\to$ Evans.

Corollary: Under the assumptions of previous theorem, if u is the unique weak sol^n to $\begin{cases} Lu = f, & U \\ u = 0, & \partial U \end{cases}$, then $\|u\|_{H^2(U)} \le C\|f\|_{L^2(U)}$
(i.e. $\|u\|_{L^2}$ is dropped).

Remarks: ① High reg. possible: if $a^{ij}, b^i, c \in C^{m+1}(\bar{U})$, $f \in H^m(U)$, $\partial U \in C^{m+2}$ and $u \in H^1_0$ a weak soln to ① then $u \in H^{m+2}(U)$ and $\|u\|_{H^{m+2}(U)} \le C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$.
② if everything C^∞, then $u \in C^\infty$.

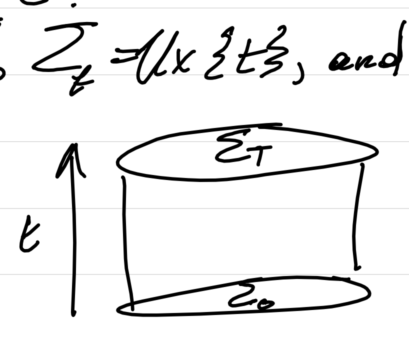
e.g. if $Lu = \lambda u$ then $(L - \lambda I)$ is unif. elliptic and $(L - \lambda I)u = f \equiv 0 \in C^\infty \Rightarrow$ e' functions $\in C^\infty$

Chapter 5 Hyperbolic eq^{ns}

Def^n A 2nd order linear PDE.
$$① \quad -\sum_{i,j=1}^{n+1} (a^{ij}(y) u_{y_i})_{y_j} + \sum_{i=1}^{n} a^i(y) u_{y_i} + a(y)u = f$$

with $y \in \mathbb{R}^{n+1}$, $a^{ij} = a^{ji}$, a^i, $a \in C^\infty(\mathbb{R}^{n+1})$ is hyperbolic if the following quadratic form:
$$q(\xi) := \sum_{i,j=1}^{n+1} a^{ij}(y) \xi_i \xi_j, \text{ the principal}$$
symbol has signature $(+, -, \dots, -)$ for all $y \in \mathbb{R}^{n+1}$, i.e. at each point y (after possibly changing basis), $q(\xi) = \lambda_{n+1} \xi_{n+1}^2 - \sum_{i=1}^{n} \lambda_i \xi_i^2$
where $\lambda_k(y) > 0 \quad \forall k = 1, \dots, n+1$.
So, by a coord. transformation, we can put ① locally in the form
$$u_{tt} - \sum_{i,j=1}^{n} (a^{ij}(x,t) u_{x_i})_{x_j} + \sum_{i=1}^{n+1} b^i(x,t) u_{x_i} + c(x,t)u = f.$$
where $(x_1, \dots, x_n, t) = (y_1, \dots, y_{n+1})$
Note, assume $\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \ge \theta |\xi|^2$, then since the coeff. of u_{tt} is 1 $(\neq 0)$, we see $\{(x,t) : t = 0\}$ is a non-characteristic surface of PDE.
In particular, could solve the PDE with analytic data $u, u_t|_{t=0}$.

5.1) Hyperbolic IBVP: Suppose $U \subset \mathbb{R}^n$ open bounded with $\partial U \in C^1$.
Define $U_T := (0,T) \times U$, $\Sigma_t = U \times \{t\}$, and $\partial^* U_T = [0,T] \times \partial U$.
$\partial U_T = \Sigma_0 \cup \Sigma_T \cup \partial^* U$ and these sets are pairwise disjoint. let $u \in C^2(U_T)$ satisfy the IBVP



$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T \\ u = \psi_0 & \text{on } \Sigma_0 \\ u_t = \psi_1 & \text{on } \Sigma_0 \end{cases} \quad \Big\} \; \frac{\text{Initial}}{\text{boundary}} \quad u = 0 \text{ on } \partial^* U_T$$

We perform an energy estimate. Multiply the PDE by u_t and integrate by parts over $U_t = (0,t) \times U$, $t \in (0,T)$.
$$0 = \int_{U_t} (u_{tt} u_t - u_t \Delta u)\, dx\, dt. \quad (\nabla \cdot (g \nabla h) = \nabla g \cdot \nabla h + g \Delta h).$$
$$= \int_{U_t} \left(\tfrac{1}{2} \partial_t (u_t)^2 - \operatorname{div}_x(u_t Du) + Du_t Du\right) dx\, dt.$$
$$= \int_{U_t} \tfrac{1}{2} \partial_t ((u_t)^2 + |Du|^2) - \operatorname{div}_x(u_t \cdot Du)\, dx\, dt$$
$$= \frac{1}{2} \int_{\Sigma_t} ((u_t)^2 + |Du|^2)\, dx - \frac{1}{2} \int_{\Sigma_0} (u_t^2 + |Du|^2)\, dx$$
$$- \int_0^t \int_{\partial U} \underbrace{u_t\, Du \cdot \vec{n}\, ds}_{=0} \quad \text{since } u|_{\partial^* U_T} = 0 \Rightarrow u_t = 0 \text{ on } \partial^* U_T.$$
$$\Rightarrow \int_{\Sigma_t} ((u_t)^2 + |Du|^2)\, dx = \int_{\Sigma_0} ((\psi_1)^2 + |D\psi_0|^2)\, dx$$

t arbitrary $\Rightarrow$ energy is conserved in time.

(A priori estimate).

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E.g.: let $v, \bar v \in C^2(U_T)$ be 2 sol^{ns} to (1) with ID $\phi_i, \bar\phi_i$. Let $u = v - \bar v$, $\psi_0 = \phi_0 - \bar\phi_0$, $\psi_1 = \phi_1 - \bar\phi_1$, then $\exists C > 0$ s.t.
$$\sup_{t\in[0,T]} \left(\|u(\cdot,t)\|^2_{H^1(\Sigma_t)} + \|u_t(\cdot,t)\|^2_{L^2(\Sigma_t)} \right) \le C \cdot \left(\|\psi_0\|^2_{H^1(\Sigma_0)} + \|\psi_1\|^2_{L^2(\Sigma_0)} \right).$$
$\longrightarrow$ uniqueness and cont. dep. on ID.

Goal: prove existence of sol^n.

Define:
$$Lu = -\sum_{i,j=1}^n (a^{ij}(x,t) u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x,t) u_{x_i} + c(x,t)\cdot u$$
with $a^{ij} = a^{ji}$, $b^i, b, c \in C^1(\bar U_T)$. Assume $\exists \theta > 0$ $\sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \ge \theta|\xi|^2$ $\forall (x,t) \in U_T$, $\xi \in \mathbb{R}^n$.

We consider the IBVP
$$\left.\begin{array}{l} u_{tt} + Lu = f \text{ in } U_T \\ u_0 = \psi_0,\ u_t = \psi_1 \text{ on } \Sigma_0 \\ u = 0 \text{ on } \partial^* U_T \end{array}\right\} \quad (2)$$

Aim: find the weak formulation

(1) Spse $u \in C^2(\bar U_T)$ and sol^n to (2).
Multiply by $v \in C^2(\bar U_T)$ s.t. $v = 0$ on $\partial^* U_T \cup \Sigma_T$ ($v \neq 0$ on Σ_0 to recover ID).
Integrate over U_T:
$$\int_{U_T} fv\,dxdt = \int_{U_T} (u_{tt}\cdot v + Lu\cdot v)\,dxdt$$
$$= \int_{U_T} (-u_t v_t + a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + b u_t v + cuv)\,dxdt + \left[\int_{\Sigma_t} u_t v\,dx \right]_{t=0}^T - \int_0^T \left(\int_{\partial\Sigma_t} a^{ij} \partial_{x_i} u\, v \cdot \vec n_i\, dS \right) dt$$
$$\Rightarrow \left.\begin{array}{l} \displaystyle\int_{U_T} (-u_t v_t + a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} \cdot v + b u_t \cdot v + cuv)\,dxdt \\ \displaystyle\quad - \int_{\Sigma_0} \psi_1(x) v(x,0)\,dx = \int_{U_T} f\cdot v\cdot dxdt \end{array}\right\} (3)$$
and $u|_{\Sigma_0} = \psi_0$, $u|_{\partial^* U_T} = 0$.

(2) Conversely (3) holds for all $v \in C^2(\bar U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$

(a) if $v \in C_c^\infty(U_T)$ then undoing the IBP, (coeffs all C^1) we get
$$0 = \int_{U_T} (u_{tt} + Lu - f)\, v\, dxdt$$
Since v arbitrary, $u_{tt} + Lu - f = 0$ on U_T.

(b) if $v \in C^\infty(\bar U_T)$ then we get
$$\int_{U_T} \underbrace{(u_{tt} + Lu - f)}_{=0 \text{ by (a)}}\, dxdt = \int_{\Sigma_0} (\psi_1 - u_t) v\, dx$$
$\Rightarrow \int_{\Sigma_0} (\psi_1 - u_t) v\,dx = 0$ $\forall v \in C^\infty(\bar U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$.
Take $v(x,t) = \chi(t)\varphi(x)$, with $\chi \in C^\infty([0,T])$ and $\varphi \in C_c^\infty(\Sigma_0)$ and also $\chi \equiv 1$ near $t=0$ and $\chi = 0$ near $t = T \Rightarrow v|_{\Sigma_0} = \varphi$ $\forall \varphi \in C_c^\infty(\Sigma_0)$.
$$\Rightarrow \int_{\Sigma_0} (\psi_1(x) - u_t(x,0))\varphi(x)\,dx = 0 \Rightarrow \psi_1 = u_t \text{ on } \Sigma_0.$$

Def^n: Spse $f \in L^2(U_T)$, $\psi_0 \in H_0^1(\Sigma_0)$, $\psi_1 \in L^2(\Sigma_0)$, $a^{ij} = a^{ji}$, $b^i, b, c \in C^1(\bar U_T)$, a^{ij} unif. elliptic. We say $u \in H^1(U_T)$ is a weak soln to the hyp IBVP (2) if $u|_{\Sigma_0} = \psi_0$, $u|_{\partial^* U_T} = 0$ (in the trace sense) and
$$\int_{U_T} (-u_t v_t + a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + b u_t v + cuv)\,dxdt - \int_{\Sigma_0} \psi_1(x) v(x,0)\,dx = \int_{U_T} fv\,dxdt \quad (3)$$
holds $\forall v \in H^1(U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$.

Theorem 8.1 A weak solution, if it exists, is unique.

Pf: if $v, \bar v$ are 2 weak sol^n to IBVP with the same ID then since the problem is linear, $u = v - \bar v$ is a weak sol^n with $f \equiv 0$, $u(x,0) = 0$, $u_t(x,0) \equiv 0$.

$$\boxed{\text{Idea: use an energy s.t. } \|u\| = 0 \Rightarrow u = 0.}$$

Want to pick $v = u_t$ (as for the wave equation) but (i) $v \notin H^1(U_T)$ since we only have $u \in H^1(U_T)$
(ii) $v \neq 0$ on Σ_T

Define $v(x,t) = \int_t^T e^{-\lambda s} u(x,s)\,ds$ some $\lambda > 0$ (pick later)
Check $v \in H^1(U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$
Also $v_t = -e^{-\lambda t} u(x,t)$. Take this v as the test function in (3) ($\psi_0 = \psi_1 = 0$) gives
$$\int_{U_T} [u_t u \cdot e^{-\lambda t} - e^{\lambda t} a^{ij} v_{t x_j} v_{x_i} + b^i u_{x_i} v + b u_t v + (c-1)uv - e^{\lambda t} v \cdot v_t]\,dxdt = 0$$
$$\Rightarrow \int_{U_T} [u_t u e^{-\lambda t} - e^{\lambda t} a^{ij} v_{t x_j} v_{x_i} + \underbrace{(b^i u v)_{x_i}}_{a} + \underbrace{(buv)_t}_{b} - (b^i_{x_i} uv + b^i u v_{x_i} + b_t uv + b u v_t) + (c-1)uv - \tfrac12 \partial_t (v^2 e^{\lambda t}) + \tfrac12 \lambda v^2 e^{\lambda t}]\,dxdt = 0$$
$\int a = 0$ since $v = u = 0$ on $\partial^* U_T$ (it's a divergence)
$\int b = 0$ since $v = 0$ on Σ_T, $u = 0$ on Σ_0.
$$\Rightarrow \left.\begin{array}{l}\displaystyle\int_{U_T} \frac12 \frac{\partial}{\partial t} (u^2 e^{-\lambda t} - a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - v^2 e^{\lambda t})\,dxdt \\ \displaystyle + \frac12 \int_{U_T} (u^2 e^{-\lambda t} + a^{ij} e^{\lambda t} v_{x_i} v_{x_j} + v^2 e^{\lambda t})\,dxdt \end{array}\right\} A$$
$$= \left.\begin{array}{l}\displaystyle\int_{U_T} \Big(-\frac12 a^{ij}_t v_{x_i} v_{x_j} e^{\lambda t} + (b^i_{x_i} + b_t + (1-c))uv \\ \qquad + b^i v_{x_i} u + b u v_t \Big)\,dxdt \end{array}\right] B$$

So
$$A = e^{-\lambda T} \int_{\Sigma_T} \underbrace{\tfrac12 u^2\,dx}_{\ge 0} + \frac{e^{\lambda T}}{2} \int_{\Sigma_0} \underbrace{(a^{ij} v_{x_i} v_{x_j} + v^2)}_{\ge 0}\,dx + \frac{\lambda}{2} \int_{U_T} (u^2 e^{-\lambda t} + e^{\lambda t} \underbrace{a^{ij} v_{x_i} v_{x_j}}_{\ge \theta|Dv|^2} + v^2 e^{\lambda t})\,dxdt$$
$$\Rightarrow A \ge \frac{\lambda}{2} \int_{U_T} (u^2 e^{-\lambda t} + \theta|Dv|^2 e^{\lambda t} + v^2 e^{\lambda t})\,dxdt$$
Also
$$B \le C\cdot(a^{ij}_t) \int_{U_T} e^{\lambda t} |Dv|^2\,dxdt + C(b, b^i_{x_i}) \int_{U_T} |u|\cdot|v|\,dxdt + C(b^i) \int_{U_T} |u|\cdot|Dv|\,dxdt + C(b) \int_{U_T} u^2 e^{-\lambda t}\,dxdt$$
(use $v_t = -e^{-\lambda t} u$)
$$\le \frac{C}{\theta} \int_{U_T} e^{\lambda t}\theta\cdot|Dv|^2 + C \int_{U_T} e^{-\lambda t}|u|^2 + e^{\lambda t}(|v|^2 + |Dv|^2)$$
$$\le C \int_{U_T} (\theta|Dv|^2 e^{\lambda t} + u^2 e^{-\lambda t} + v^2 e^{\lambda t})\,dxdt$$

Now $|A| = |B|$
$$\Rightarrow \left(\frac{\lambda}{2} - C\right) \int_{U_T} \underbrace{(u^2 e^{-\lambda t} + \theta|Dv|^2 e^{\lambda t} + v^2 e^{\lambda t})}_{\ge 0}\,dxdt \le 0$$
Pick $\lambda > 2C \Rightarrow \int_{U_T} e^{-\lambda t} u^2\,dxdt = 0$
$\Rightarrow u = 0$ a.e. on U_T.

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Theorem 5.2 (Existence of Sol^n)
Given $\psi_0 \in H_0^1(\mathcal{U})$, $\psi_1 \in L^2(\mathcal{U})$, $f \in L^2(\mathcal{U}_T)$ then $\exists!$ weak sol^n $u \in H^1(\mathcal{U}_T)$ of (2) with
$$\|u\|_{H^1(\mathcal{U}_T)} \leq C \cdot \left(\|\psi_0\|_{H^1(\mathcal{U})} + \|\psi_1\|_{H^1(\mathcal{U})} + \|f\|_{L^2(\mathcal{U}_T)} \right)$$
$(\mathcal{U} = \Sigma_0)$.

Pf: (Galerkin's Method)
Idea: project everything onto the finite-dim subspace of L^2 spanned by the first N eigenfunctions of the Dirichlet Laplacian. Take $N \to \infty$.

Proof (1): Recall the e-f^{ns}: $\{\varphi_k\}_{k=1}^{\infty}$ of $L = -\Delta$ with Dirichlet BCs form an orthonormal basis of $L^2(\mathcal{U})$. Have $\varphi_k \in H_0^1(\mathcal{U})$ and by elliptic regularity $\varphi_k \in C^\infty(\overline{\mathcal{U}})$ (provided $\partial\mathcal{U}$ is smooth).
Recall $(\varphi_k, \varphi_l)_{L^2(\mathcal{U})} = \delta_{kl}$ and if $u \in L^2(\mathcal{U})$ then $u = \sum_{k=1}^{\infty} (u, \varphi_k)\varphi_k$ with convergence in $L^2(\mathcal{U})$.

(2) Finite-dim approx: First consider $\psi_0, \psi_1 \in C_c^\infty(\mathcal{U})$, $f \in C_c^\infty(\mathcal{U}_T)$. These spaces are dense in $H_0^1(\mathcal{U})$, $L^2(\mathcal{U})$, $L^2(\mathcal{U}_T)$.
Define $u^N(x,t) = \sum_{k=1}^{N} u_k(t)\varphi_k(x)$.
Assume each $u_k(t) \in C^2([0,T])$ and suppose that $u^N(x,t)$ is a weak sol^n to equation (2).
Taking $v(x,t) = \rho(t)\varphi_l(x)$ a test function with $\rho \in C_c^\infty((0,T))$ arbitrary in (2)
$$\Rightarrow \int_{\mathcal{U}_T} \left(-u_t^N \dot{\rho}\varphi_l + a^{ij} u_{x_j}^N (\varphi_l)_{x_i}\rho + b^i u_{x_i}^N \rho\varphi_l + b(u^N_t)\rho\varphi_l + c u \rho \varphi_l - f\rho\varphi_l \right) dxdt = 0.$$

Note $\int_{\mathcal{U}_T} -(u^N)_t \dot{\rho}\varphi_l \, dxdt$
$$= \int_{\mathcal{U}_T} -(u^N)_{tt}\, \rho\, \varphi_l \, dxdt$$
i.e. our identity looks like
$$\int_0^t \int_{\Sigma_t} Q(x,t)\rho(t)\,dx\,dt = 0 \quad \forall \rho$$
$$\Rightarrow \int_{\Sigma_t} Q(x,t)\,dx = 0.$$

$$\Rightarrow \text{(4)} \begin{cases} (u^N_{tt}, \varphi_l)_{L^2(\Sigma_t)} + \int_{\Sigma_t} \big(a^{ij} u^N_{x_j} (\varphi_l)_{x_i} \\ \quad + b^i (u^N)_{x_i} \varphi_l + b u^N_t \varphi_l + c u^N \varphi_l \big)\, dx \\ \quad = (f, \varphi_l)_{L^2(\Sigma_t)} \end{cases}$$
and (4) holds for every $t \in [0,T]$, each $l = 1, \dots, N$.

By orthonormality, $(u^N_{tt}, \varphi_l)_{L^2(\Sigma_t)}$
$$= \sum_{k=1}^{N} (\ddot{u}_k \varphi_k, \varphi_l)_{L^2(\Sigma_t)} = \ddot{u}_l(t)$$

In this way, we get for $l = 1, \dots, N$
$$\ddot{u}_l(t) + \sum_{k=1}^{N} \left(\alpha_{l,k}(t) u_k(t) + \beta_{l,k}(t)\dot{u}_k(t) \right) = f_l(t)$$
where $\alpha_{lk}(t) = \int_{\Sigma_t} \big(a^{ij}(\varphi_l)_{x_j}(\varphi_k)_{x_i} + b^i(\varphi_l)_{x_i}\varphi_k + c\varphi_l\varphi_k \big)\,dx$
$$\beta_{l,k}(t) = \int_{\Sigma_t} b(x,t)\varphi_l\varphi_k\,dx$$
$$f_l(t) = \int_{\Sigma_t} f(x,t)\varphi_l(x)\,dx$$
and $u_l(0) = (\psi_0, \varphi_l)_{L^2(\Sigma_0)}$, $\dot{u}_l(0) = (\psi_1, \varphi_l)_{L^2(\Sigma_0)}$.

This is a system of N second order ODEs, linear in u_k, with coeffs that are bounded uniformly in C^1 for $t \in [0,T]$ $\Rightarrow$ Picard-Lindelöf $\exists!$ sol^n $u_k(t) \in C^2([0,T])$ and also $u^N \in H^1(\mathcal{U}_T)$, $\partial_t u^N \in H^1(\mathcal{U}_T)$

(3) Want uniform estimates $\|u^N\|_{H^1(\mathcal{U}_T)} \leq C$ indep of N.
Multiply (4) by $e^{-\lambda t}\dot{u}_l(t)$, $\lambda > 0$, sum over $l = 1, \dots, N$, and integrate over $[0,\tau]$, $\tau \in [0,T]$.
eg.:
$$\sum_{l=1}^{N} \int_0^\tau e^{-\lambda t}\dot{u}_l(t) \int_{\Sigma_t} u^N_{tt}\varphi_l \, dxdt = \int_{\mathcal{U}_\tau} e^{-\lambda t} u^N_{tt} u^N_t \, dxdt$$
We find
$$\int_{\mathcal{U}_\tau} \Big[(u^N)_{tt} \times (u^N)_t + a^{ij}(u^N)_{x_i}(u^N)_{tx_j} + b^i (u^N)_{x_i}(u^N)_t + b (u^N)_t^2 + c u^N (u^N)_t \Big] e^{-\lambda t}\,dxdt$$
$$= \int_{\mathcal{U}_T} f (u^N)_t e^{-\lambda t}\,dxdt$$
Similar to the proof of uniqueness, rearrange this as
$$\tilde{A}\left\{ \int_{\mathcal{U}_T} \frac{1}{2}\frac{d}{dt}\left(Q_a e^{-\lambda t} \right) dxdt + \frac{\lambda}{2}\int_{\mathcal{U}_T} Q_a e^{-\lambda t}\,dxdt \right.$$
$$\tilde{B}\left\{ = \int_{\mathcal{U}_\tau} \Big[\frac{1}{2}(a^{ij})_t (u^N)_{x_i}(u^N)_{x_j} - b^i (u^N)_t (u^N)_{x_i} - b\cdot(u^N)_t^2 + (1-c) u^N u^N_t + f (u^N)_t \Big] e^{-\lambda t}\,dxdt \right.$$

$Q_a = (u^N)_t^2 + a^{ij}(u^N)_{x_i}(u^N)_{x_j} + (u^N)^2$
let $Q_\theta = (u^N)_t^2 + \theta |Du^N|^2 + |u^N|^2$.
Using uniform ellipticity, Young's, $e^{-\lambda t} \leq 1$, etc, we get
$$\tilde{B} \leq C \int_{\mathcal{U}_\tau} Q_\theta e^{-\lambda t}\,dxdt + \|f\|^2_{L^2(\mathcal{U}_\tau)}$$
$$\tilde{A} \geq \frac{e^{-\lambda\tau}}{2} \int_{\Sigma_\tau} Q_\theta\,dx - \frac{1}{2}\int_{\Sigma_0} Q_a\,dx + \frac{\lambda}{2}\int_{\mathcal{U}_\tau} Q_\theta e^{-\lambda t}\,dxdt$$

Use $|\tilde{A}| = |\tilde{B}|$, for $\frac{\lambda}{2} - C \geq \frac{1}{2}$ we get
$$e^{-\lambda\tau}\int_{\Sigma_\tau} Q_\theta\,dx + \int_0^\tau \int_{\Sigma_t} Q_\theta e^{-\lambda t}\,dxdt$$
$$\leq \int_{\Sigma_0} Q_a\,dx + C\cdot\|f\|^2_{L^2(\mathcal{U}_\tau)}$$
$$\leq C\cdot\left(\|u^N(\cdot,0)\|^2_{H^1(\Sigma_0)} + \|\dot{u}^N(\cdot,0)\|^2_{L^2(\Sigma_0)} + \|f\|^2_{L^2(\mathcal{U}_T)} \right)$$
true for all $\tau \in [0,T]$. RHS is independent of τ, also use $e^{-\lambda\tau} \geq e^{-\lambda T}$ for $\tau \in [0,T]$.

$$\Rightarrow \sup_{\tau\in[0,T]} \left(\|u^N(\cdot,\tau)\|^2_{H^1(\Sigma_\tau)} + \|\dot{u}^N(\cdot,\tau)\|^2_{L^2(\Sigma_\tau)} \right)$$
$$+ \underbrace{\int_0^T \left(\|u^N(\cdot,t)\|^2_{H^1(\Sigma_t)} + \|\dot{u}^N(\cdot,t)\|^2_{L^2(\Sigma_t)} \right) dt}_{\|u^N\|^2_{H^1(\mathcal{U}_T)}}$$
$$\leq \underset{\substack{\| \\ C(T, a^{ij}, b^i, b, c)}}{C} \cdot e^{\lambda T}\left(\|u^N(\cdot,0)\|^2_{H^1(\Sigma_0)} + \|\dot{u}^N(\cdot,0)\|^2_{L^2(\Sigma_0)} + \|f\|^2_{L^2(\mathcal{U}_T)} \right)$$
Since $u^N(0) = \sum_{k=1}^{N} (\psi_0, \varphi_k)\varphi_k \xrightarrow{N\to\infty} \psi_0$ in $H^1(\Sigma_0)$.

If $\psi_0 \neq 0$ then for large N,
$$\|u^N(0)\|_{H^1(\Sigma_0)} \leq 2\|\psi_0\|_{H^1(\Sigma_0)}$$
Similarly $\|\dot{u}\|_{L^2(\Sigma_0)} \leq 2\cdot\|\psi_1\|_{L^2(\Sigma_0)}$.

$$\Rightarrow \|u^N\|_{H^1(\mathcal{U}_T)} \leq \underbrace{C\left(\|\psi_0\|_{H^1(\Sigma_0)} + \|\psi_1\|_{H^1(\Sigma_0)} + \|f\|_{L^2(\mathcal{U}_T)} \right)}_{= C_1 \text{ indep of } N.}$$

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<u>Proof:</u> Construct $u^N(x,t) := \sum_{\ell=1}^{N} u_\ell(t)\varphi_\ell(x)$ with φ_ℓ e'functions and $u_\ell(t) \in C^2([0,T])$ determine from the ODE

$$\ddot{u}_\ell(t) + \sum\left(\alpha_{\ell,k}(t)u_k(t) + \beta_{\ell,k}(t)\dot{u}_k(t)\right) = 0$$

$$u_\ell(0) = (\psi_0, \varphi_\ell)_{L^2(\Sigma_0)}, \quad \dot{u}_\ell(0) = (\psi_1, \varphi_\ell)_{L^2(\Sigma_0)}.$$

These ODEs come from projecting ② onto span $\{\varphi_1, \dots, \varphi_N\}$. We showed

$$\|u^N\|_{H^1(U_T)} \le C_1 = C\left(\|\psi_0\|_{H^1(\Sigma_0)} + \|\psi_1\|_{L^2(\Sigma_0)} + \|f\|_{L^2(U_T)}\right)$$

Note $u^N \in H^1_\partial(U_T) := \{\phi \in H^1 : \phi|_{\partial^* U_T} = 0\}$ is a closed subspace of $H^1(U_T)$

$\Rightarrow$ weakly sequentially compact (bounded sets)

$\Rightarrow \exists\, (u^{N_i})_i$ s.t. $u^{N_i} \rightharpoonup u$ in $H^1_\partial(U_T)$ for some $u \in H^1_\partial(U_T)$.

Also $\|u\|_{H^1(U_T)} \le \liminf_{i\to\infty} \|u^{N_i}\|_{H^1(U_T)} \le C_1$.

④ Want to show that u is desired weak soln. Relabel $u^{N_i} \to u$. Fix $m < N$. Consider

$$v = \sum_{k=1}^{m} v_k(t)\cdot\varphi_k(x) \text{ with } v_k \in H^1((0,T))$$

and $v_k(T) = 0$. Note v is a test function for the weak formulation. Recall

④ $$(u^N_{tt}, \varphi_k)_{L^2(\Sigma_t)} + \int_{\Sigma_t}\Big(a^{ij}u^N_{x_j}(\varphi_k)_{x_i} + b^i(u^N)_{x_i}\varphi_k + bu^N_t\varphi_k + cu^N\varphi_k\Big)dx = (f, \varphi_k)_{L^2(\Sigma_t)}; \; k = 1, \dots, N.$$

Multiply the k^{th} eqn in ④ by $v_k(t)$ and sum over $k = 1, \dots, N$ ($v_k(t) = 0$, $k = m+1, \dots, N$)

$$\Rightarrow (u^N_{tt}, v)_{L^2(\Sigma_t)} + \int_{\Sigma_t}\Big(a^{ij}(u^N)_{x_j}v_{x_i} + b^i(u^N)_{x_i}v + bu^N_t v + cuv\Big)dx.$$

Integrate over $[0,T]$, IBP, use $v(T) = 0$.

$$\Rightarrow -\int_{\Sigma_0} u^N_t v\,dx + \int_{U_T}\Big(u^N_t v_t + a^{ij}u^N_{x_j}v_{x_i} + b^i(u^N)_{x_i}v + bu^N_t v + cuv\Big)dxdt = \int_{U_T} f\cdot v\,dxdt$$

Since $N > m$, $\int_{\Sigma_0}(u^N_t)v\,dx \overset{\text{Parseval}}{=} \int_{\Sigma_0}\psi_1\cdot v\,dx$

Pass to weak limit $\Longrightarrow$ (†) $=$

$$-\int_{\Sigma_0}\psi_1 v\,dx + \int_{U_T}\Big(-u_t v_t + a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i}v + b\,u_t v + cuv\Big)dxdt = \int_{U_T} fv\,dxdt$$

i.e. for <u>these</u> v's, u is a weak sol^{n}

<u>Exercise:</u> the linear space

$$\left\{v = \sum_{k=1}^{m}\varphi_k(x)v_k(t),\ v_k \in H^1([0,T]),\ v_k(T) = 0,\ m = 1, 2, \dots\right\}$$

is dense in $H^1_0(U_T)$ and so (†) holds $\forall v \in H^1_\partial(U_T)$.

⑤ Remains to prove: $u|_{\Sigma_0} = \psi_0$. (trace sense)

For each fixed $k = 1, 2, \dots$,

$$\Phi_k : H^1(U_T) \to \mathbb{R}$$
$$w \mapsto \int_{\Sigma_0} w\varphi_k\,dx \text{ is a bdd lin. map.}$$

To check this:

$$|\Phi_k(w)| \le \int_{\Sigma_0}|w\varphi_k|\,dx \le \|w\|_{L^2(\Sigma_0)}\underbrace{\|\varphi_k\|_{L^2(\Sigma_0)}}_{=1}$$

$$\le \|w\|_{L^2(\partial U_T)} \underset{\text{Trace theorem}}{\le} C\cdot\|w\|_{H^1(U_T)}.$$

By the weak convergence, $\Phi_k(u^N) \to \Phi_k(u)$ for $k = 1, \dots$

$$\Rightarrow \int_{\Sigma_0}\psi_0\varphi_k\,dx \overset{\text{Parseval.}}{=} \int_{\Sigma_0}u^N(x,0)\varphi_k\,dx \to \int_{\Sigma_0}u(x,0)\varphi_k\,dx$$

$$\Rightarrow \int_{\Sigma_0}(\psi_0 - u(x,0))\varphi_k\,dx = 0 \quad \forall k$$

$$\Rightarrow \psi_0 = u(x,0) \text{ on } \Sigma_0 \qquad \square$$

<u>Def^n:</u> if X a Banach space, the <u>Bochner space</u> $L^p((0,T);X)$ is defined by

$$L^p((0,T);X) = \left\{u : (0,T) \to X : \|u\|_{L^p((0,T);X)} < \infty\right\}$$

where $\|u\|_{L^p((0,T);X)} := \begin{cases}\left(\int_0^T \|u\|_X^p\,dt\right)^{1/p}, & 1 \le p < \infty \\ \operatorname{ess\,sup}_{t\in(0,T)} \|u(t)\|_X, & p = \infty\end{cases}$

<u>Remark:</u> In step ③ we showed $\|u\|_{H^1(U_T)} \le C_1$. In fact, the weak sol^{n} satisfies $\|u_t\|_{L^\infty((0,T);L^2(U))} + \|u\|_{L^\infty((0,T);H^1(U))} \le C_1$ and instead of $u \in H^1(U_T)$ ($\int_0^T |\ |^2 dt$), can conclude $u \in L^\infty\underbrace{((0,T); H^1(U))}_{\sup_t |\cdot|}$

5.3 Finite Speed of propagation

<u>Information can only travel at a finite speed.</u>

<u>Def^n</u> Let $\Sigma \subset \mathbb{R}^{n+1}$ be a zero-set of some function F, i.e. $\Sigma = \{(x,t) \in \mathbb{R}^{n+1} : F(x,t) = 0\}$

Define $w(F_x, \dots, F_t) = (F_t)^2 - \sum_{i,j=1}^{n} a^{ij}F_{x_i}F_{x_j}$

Say Σ is $\begin{cases}\text{space-like} \\ \text{time-like} \\ \text{null}\end{cases}$ if $\begin{cases}w > 0 \\ w < 0 \\ w = 0\end{cases}$

Eg ① plane $\{t = 0\}$ is spacelike. $F(x,t) = t$.

② Cylinder $F = |x - x_0|^2 - R^2$ is timelike

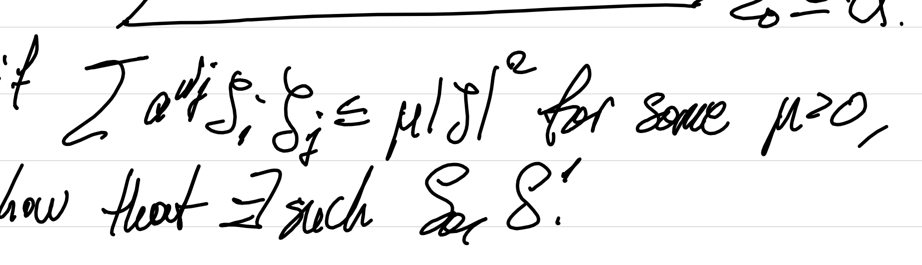
Let $S_0 \subset U$ be an open set with $\partial S_0 \in C^\infty$. Let $\tau : S_0 \to (0,T)$ be smooth f^n s.t. $\tau|_{\partial S_0} = 0$.

Let $S' := \operatorname{graph}(\tau) = \{(x, \tau(x)) : x \in S_0\}$.

If $F(x_1, \dots, x_n, t) = t - \tau(x)$, then we see S' is spacelike if $1 - \sum_{i,j=1}^{n} a^{ij}\tau_{x_i}\tau_{x_j} > 0$

$$\Longleftrightarrow \sum_{i,j=1}^{n} a^{ij}(x)\tau_{x_i}\tau_{x_j} < 1 \ \forall x \in S_0.$$

let $D = \{(x,t) \in U_T : x \in S_0,\ 0 < t < \tau(x)\}$



<u>Ex:</u> if $\sum a^{ij}\xi_i\xi_j \le \mu|\xi|^2$ for some $\mu > 0$, can show that $\exists$ such S_0, S'.

<u>Theorem:</u> (Domain of dependence) If S' space-like and u a weak sol^{n} to ② then $u|_D$ depends only on $\psi_0|_{S_0}$, $\psi_1|_{S_0}$ and $f|_D$.

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Proof: (Uniqueness proof Thm 5.1) By linearity, it suffices to prove $u|_D = 0$ if $\psi_0|_{S_0} = 0$, $\psi_1|_{S_0} = 0$ and $f|_D = 0$. Take test function

$$V(x,t) = \begin{cases} \int_t^{\tau(x)} e^{-\lambda s} u(x,s)\,ds, & (x,t) \in D \\ 0 & \text{otherwise} \end{cases}$$

Ex: check $v \in H^1(U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$ and $V_{x_i} = \tau_{x_i} e^{-\lambda\tau(x)} u(x,\tau(x)) + \int_t^{\tau(x)} e^{-\lambda s} u_{x_i}(x,s)\,ds$

in D, $v_t = -e^{-\lambda t} u(x,t)$ in D and $v_{x_i} = v_t = 0$ on $U_T \setminus D$. Insert into def^n of weak sol^n.

$$\left.\int_D \tfrac{1}{2}\partial_t\left(u^2 e^{-\lambda t} - a^{ij} V_{x_i} V_{x_j} e^{\lambda t} - v^2 e^{\lambda t}\right) dx\,dt \right] \bar{A}$$
$$+ \frac{\lambda}{2}\int_D \left(u^2 e^{-\lambda t} + a^{ij} V_{x_i} V_{x_j} e^{\lambda t} + v^2 e^{\lambda t}\right) dx\,dt$$
$$= \int_D \Big(\tfrac{1}{2} a^{ij}_t V_{x_i} V_{x_j} e^{\lambda t} + (b^i_{x_i} + b_t + 1 - c) u v$$
$$+ b^i v_{x_i} u + b u v_t\Big)\,dx\,dt$$

By Fubini, $\int_D \cdot \,dx\,dt = \int_{S_0} dx \left(\int_0^{\tau(x)} \cdot\, dt\right)$,

using that $v|_{S} = 0$ and $V_{x_i}|_{S} = \tau_{x_j} u(x,\tau(x)) e^{-\lambda\tau(x)}$

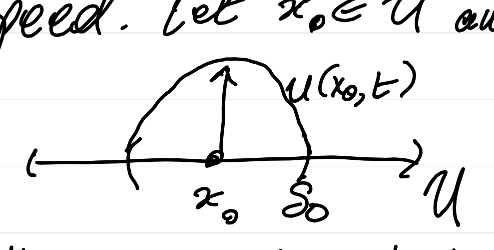
$$\Rightarrow \bar{A} = \frac{1}{2}\int_{S_0} \underbrace{u^2(x,\tau(x))}_{\geq 0} \underbrace{e^{-\lambda\tau(x)}}_{\geq 0} \big(\underbrace{1 - a^{ij}\tau_{x_i}\tau_{x_j}}_{\geq 0}\big)\,dx$$
$$+ \frac{1}{2}\int_{S_0} \big(\underbrace{a^{ij} V_{x_i} V_{x_j}}_{\geq 0} + \underbrace{v^2}_{\geq 0}\big)\Big|_{t=0}\,dx$$

Continue as in Th 5.1.

$$\Rightarrow \left(\frac{\lambda}{2} - C\right)\int_D \big(\underbrace{u^2 e^{-\lambda t}}_{\geq 0} + \underbrace{\theta |Du|^2 e^{\lambda t}}_{\geq 0} + \underbrace{v^2 e^{\lambda t}}_{\geq 0}\big)\,dx\,dt \leq 0.$$

If λ large, then we get $u|_D = 0$.

Remark: no signal can travel faster than some fixed speed. Let $x_0 \in U$ and S_0 some ball about x_0.



if $(x_0,t) \in D$ then any data outside S_0 does not influence $u(x_0,t)$. Only after some time $t > \tau(x_0)$ will the function be determined by data outside S_0. $\Rightarrow$ everything is local in hyperbolic PDE.

5.4: Hyperbolic Regularity

We have shown existence, uniqueness of weak sol^n to $u_{tt} + Lu = f$ (with IC, BCs). Given $\psi_0 \in H_0^1(U)$, $\psi_1 \in L^2(U)$, $f \in L^2(U_T)$. We have shown

$$\|u\|_{L_t^\infty H_0^1(U)} + \|u_t\|_{L_t^\infty L_x^2} + \|u\|_{H^1(U_T)}$$
$$\leq C\left(\|\psi_0\|_{H^1(U)} + \|\psi_1\|_{L^2(U)} + \|f\|_{L^2(U_T)}\right).$$

No gain in x-regularity. No gain in t-reg.

Example: suppose $u \in C^\infty(\bar{U}_T)$ solves

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T \\ u = \psi_0, u_t = \psi_1 & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases}$$

set $w = u_t \Rightarrow \begin{cases} w_{tt} - \Delta w = 0, & U_T \\ w = \psi_1, w_t = \Delta\psi_0, & \Sigma_0 \\ w = 0, & \partial^* U_T. \end{cases}$

$$\Rightarrow \|w\|_{L_t^\infty H_x^1} + \|w_t\|_{L_t^\infty L^2} + \|w\|_{H^1(U_T)}$$
$$\leq C \cdot \left(\|\psi_1\|_{H^1(U)} + \|\Delta\psi_0\|_{L^2(U)}\right).$$

i.e., control u_{tt} and u_{xt} in $L^2(U)$ in terms of initial data.

To control $u_{x_i x_j}$ use elliptic regularity:

$$\|u\|_{H^2(U)} \underset{\text{elliptic}}{\leq} C \cdot \|\Delta u\|_{L^2(U)} \underset{\text{PDE}}{=} C \|u_{tt}\|_{L^2(U)}$$

All together: $\|u\|_{L_t^\infty H_x^2} + \|u_t\|_{L_t^\infty H_x^1} + \|u_{tt}\|_{L_t^\infty L_x^2}$

$$\leq C \cdot \left(\|\psi_0\|_{H^2(\Sigma_0)} + \|\psi_1\|_{H^1(\Sigma_0)}\right).$$

Th^m 5.4: (Hyp. Regularity)

Spse $a^{ij}, b^i, c \in C^2(\bar{U}_T)$, $\partial U \in C^2$, then for $\psi_0 \in H^2(U) \cap H_0^1(U)$, $\psi_1 \in H_0^1(U)$, $f, f_t \in L^2(U_T)$ the unique weak sol^n to (2) satisfies:

$u \in H^2(U_T) \cap L_t^\infty H_x^2$, $u_t \in L_t^\infty H_0^1$, $u_{tt} \in L_t^\infty L_x^2$.

Proof (1) by approx, $f \in C^\infty(\bar{U}_T)$, $\psi_0, \psi_1 \in C_c^\infty(U)$. Use $u^N(x,t) = \sum_{k=1}^{N} u_k(t)\varphi_k(x)$. Derived ODE for $u_k(t)$.

Now, coeffs of ODE are $C^2([0,T])$

$\Rightarrow u_k \in C^3([0,T])$

(2) Since $u^N \in C_t^3$ differentiate (4) wrt t.

$$(u^N_{ttt}, \varphi_k)_{L^2(U)} + \int_{\Sigma_t} \Big(a^{ij}(u^N)_{tx_j}(\varphi_k)_{x_j}$$
$$+ b_i (u^N)_{tx_i}\varphi_k + b(u^N)_{tt}\varphi_k + c u^N_t \varphi_k\Big)\,dx$$
$$= (f_t, \varphi_k)_{L^2(\Sigma_t)} - \int_{\Sigma_t}\Big(a^{ij}_t u^N_{x_i}(\varphi_k)_{x_j} +$$
$$b^i_t u^N_{x_i}\varphi_k + b_t u^N_t \varphi_k + c_t u^N \varphi_k\Big)\,dx.$$

Multiply by $\ddot{u}_k e^{-\lambda t}$, sum over $k = 1, \dots, N$, $\int_0^\tau dt$, $\tau \in [0,1]$.

$$\Rightarrow \sup_{t \in [0,\tau]}\left(\|u^N_t\|^2_{H^1(\Sigma_t)} + \|u^N_{tt}\|^2_{L^2(\Sigma_t)}\right)$$
$$+ \|u_t\|^2_{H^1(U_T)}$$
$$\leq e^{\lambda T} C \cdot \Big(\|\psi_0\|^2_{H^1(\Sigma_0)} + \|\psi_1\|^2_{L^2(\Sigma_0)} + \|f\|^2_{L^2(U_T)}$$
$$+ \underbrace{\|u^N_t\|^2_{H^1(\Sigma_0)}}_{(a)} + \underbrace{\|u^N_{tt}\|^2_{L^2(\Sigma_0)}}_{(b)} + \|f_t\|^2_{L^2(U_T)}\Big).$$

So (a) $\|u^N_t\|_{H^1(\Sigma_0)} \leq C \cdot \|\psi_1\|_{H^1(\Sigma_0)}$

For (b), use eq^n (4) on the initial slice $t = 0$. Multiply (4) by $\ddot{u}_k$, sum $k = 1, \dots, N$.

$$\Rightarrow \|u^N_{tt}\|^2_{L^2(\Sigma_0)} = -\int_{\Sigma_0}\Big(a^{ij}\underbrace{(u^N)_{x_i}(u^N)_{ttx_j}}_{(*)}$$
$$+ b^i (u^N)_{x_i}(u^N)_{tt} + b u^N_t u^N_{tt} + c u^N u^N_{tt}\Big)\,dx$$
$$+ (f, u^N_{tt})_{L^2(\Sigma_0)}$$

IBP on $(*)$ $= \int_{\Sigma_0}(a^{ij}u^N_{x_j})_{x_i} u^N_{tt}$ + "everything else"

By C-S: $\|u^N_{tt}\|_{L^2(\Sigma_0)} \leq C \cdot \Big(\|u^N\|_{H^2(\Sigma_0)} + \|u^N_t\|_{L^2(\Sigma_0)}$
$$+ \underbrace{\|f\|_{L^2(\Sigma_0)}}\Big).$$

Exercise to control $\|f\|_{L^2(\Sigma_0)} \leq \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)}$.

(3) Control $\|u^N\|_{H^2(\Sigma_0)}$ unif. in N.

$$(\Delta u^N, \Delta u^N)_{L^2(\Sigma_0)} = (u^N, \Delta^2 u^N)_{L^2(\Sigma_0)}$$
$$= (\psi_0, \Delta^2 u^N)_{L^2(\Sigma_0)} = (\Delta\psi_0, \Delta u^N)_{L^2(\Sigma_0)}.$$

($\psi_0 = 0 = \Delta\varphi_k = \varphi_k$ on $\partial\Sigma_0$.)

$$\Rightarrow \|\Delta u^N\|_{L^2(\Sigma_0)} \leq \|\Delta\psi_0\|_{L^2(\Sigma_0)} \leq \|\psi_0\|_{H^2(\Sigma_0)}$$

With elliptic regularity,

$$\|u^N\|_{H^2(\Sigma_0)} \lesssim \|\psi_0\|_{H^2(\Sigma_0)}.$$

Summary: $\|u^N_t\|_{L_t^\infty H_x^1} + \|u^N_{tt}\|_{L_t^\infty L_x^2} + \|u^N_t\|_{H^1(U_T)}$

$$\leq C \cdot \underbrace{\Big(\|\psi_0\|_{H^2(\Sigma_0)} + \|\psi_1\|_{H^1(\Sigma_0)} + \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)}\Big)}_{C_2}.$$

By Banach-Alaoglu, $u_t \in H^1(U_T)$, $u_{tt} \in L_t^\infty L_x^2$, $u_{tt} \in L_t^\infty L_x^2$. For spatial derivatives, write $Lu = f - u_{tt}$. By elliptic regularity on Σ_t (for a.e. t)

$$\|u\|_{H_x^2} \leq \|f\|_{L_x^2} + \|u_{tt}\|_{L_x^2} + \|u\|_{L_x^2}$$
$$\leq C \cdot C_2$$

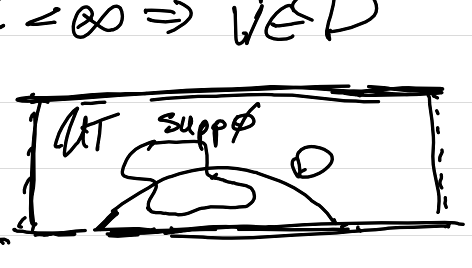
$\Rightarrow u \in L_t^\infty H_x^2$.

Sketch for why $v(x,t) = \begin{cases} \int_t^{\tau(x)} e^{-\lambda s} u(x,s)\,ds, & (x,t) \in D \\ 0, & \text{otherwise} \end{cases}$

is in $H^1(U_T)$. On compact sets $V \subset\subset D$, the difference quotients $\Delta_i^h v$, $i = x_1, \dots, x_n, t$ are bdd in their L^2 norm (indep. of V).

Hence $\sup_{V \subset\subset D}\|Dv\| \leq C < \infty \Rightarrow v \in D$

Thus, $v(x,t) \in H^1(D)$ and then extend v by zero to U_T.



ANALYSES OF PDE

LECTURE 24

Recap: Hyperbolic eqns:
- Uniqueness $\to$ energy method $(v \simeq u_t)$
- Existence $\to$ Galerkin's method $(\ldots e^{\lambda T} \ldots)$
- $\to$ to allow $T=\infty$, on U unbounded can use Hille-Yoshida thm (Brezis PDE + Sobolev space text book).

Hyperbolic regularity, if $\psi_0 \in H^2(U) \cap H_0^1(U)$, $\psi_1 \in H_0^1(U)$ and $f, f_t \in H^1(U_T)$ then weak sol^n $u \in H^2(U_T)$ and also

$$\|u\|_{L_t^\infty H^2(U)} + \|u_t\|_{L_t^\infty H_0^1(U)} \le C \cdot \left(\|\psi_0\|_{H^2} + \|\psi_1\|_{H^1} + \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)} \right).$$

Averaging effect of Laplacian:

Consider $u: \mathbb{R} \to \mathbb{R}$, $h > 0$.
Average value of u on $[-h, h]$ is $\bar{u} = \frac{1}{2h} \int_{-h}^{h} u(x)\,dx$. Taylor expand:

$$u(x) = u(0) + u'(0)x + u''(0)\frac{x^2}{2!} + O(x^3)$$

$$\Rightarrow \bar{u} = u(0) + \frac{u''(0)}{6} h^2 + O(h^4)$$

$$\Rightarrow \Delta u|_{x=0} = u''(0) = \frac{6}{h^2}(\bar{u} - u(0)) + O(h^2).$$

The Laplacian measures the difference from the average over nearby points. This generalises $\Delta u|_p = \lim_{r \to 0^+} \frac{2n}{r^2} \frac{1}{\omega(S_r)} \int_{S_r} (u(x) - u(p))\,dx$,
S_r = sphere of radius r around p.
$\to$ Mean Value Property for harmonic f^{ns}.
Consider the heat eq^n: $u_t = \Delta u$. $\to$ if average is hotter than at the point p itself $(\bar{u} > u(p))$ then $\partial_t u|_p > 0$, i.e., temp will rise at p.

Consider
$$\begin{cases} u_t - \Delta u = f & \text{on } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases}$$

Multiply PDE by u: $\frac{1}{2}\partial_t(u^2) - \mathrm{div}_x(u Du) + |Du|^2 = fu.$

Integrate over $[0,t] \times U$.

$$\Rightarrow \frac{1}{2}\int_{\Sigma_t} u^2\,dx + \int_{U_t} |Du|^2\,dx\,dt = \int_{U_t} uf\,dx\,dt + \int_{\Sigma_0} \psi^2\,dx.$$

Young's inequality: $\int_{U_t} uf \le \varepsilon \underbrace{\int_{U_t} u^2\,dx\,dt}_{\le C \cdot \int |Du|^2 \text{ by Poincaré.}} + \frac{4}{\varepsilon}\int_{U_t} f^2$

All together,
$$\underbrace{\int_{\Sigma_t} u^2\,dx}_{\text{"energy"}} + \int_{U_t} \underbrace{(u^2 + |Du|^2)}_{\ge 0}\,dx\,dt \le C\Big(\int_{U_t} f^2\,dx\,dt + \underbrace{\int_{\Sigma_0} \psi^2\,dx}_{\text{energy at } t=0}\Big).$$

Energy not conserved but decreasing. Take sup over $t \in [0, T]$:

$$\|u\|_{L_t^\infty L^2(U)} + \|u\|_{L_t^2 H^1(U)} \le C \cdot \left(\|f\|_{L^2(U_T)} + \|\psi\|_{L^2(\Sigma_0)} \right)$$

In Sheet 4, apply this to parabolic eqns. $u_t + Lu = f$.
You'll show weak sol^{ns} exist (Galerkin Method), unique (energy method).
For regularity, assume we have a smooth sol^n to the heat eq^n:
Multiply the PDE by $u_t \Rightarrow u_t^2 - \mathrm{div}(u_t Du) + \frac{1}{2}\partial_t |Du|^2 = u_t f$.
Young: $\frac{1}{2}u_t^2 + \frac{1}{2}\partial_t |Du|^2 \le \frac{1}{2}f^2 + \mathrm{div}(u_t Du)$.
Integrate over $U_t = [0,t] \times U$: $\to$ check bdry term

$$\frac{1}{2}\int_{U_t} u_t^2\,dx\,dt + \frac{1}{2}\int_{\Sigma_t} |Du|^2\,dx \le \frac{1}{2}\int_{U_t} f^2\,dx\,dt + \frac{1}{2}\int_{\Sigma_0} |Du|^2\,dx$$

Take sup $t \in [0, T]$ we get:

$$\|u_t\|_{L^2(U_T)} + \|Du\|_{L_t^\infty L^2(U)} \le C\left(\|f\|_{L^2(U_T)} + \|\psi\|_{H^1(\Sigma_0)} \right)$$

Use the PDE: $-\Delta u = f - u_t$ at each t, $u(t, \cdot) = 0$ on ∂U. By elliptic estimates

$$\|u\|_{H^2(U)} \le \|\Delta u\|_{L^2(U)} \le \|f\|_{L^2(U)} + \|u_t\|_{L^2(U)}.$$

Integrate in time

$$\|u\|_{L_t^2 H^2(U)} \le C \cdot \left(\|f\|_{L^2(U_T)} + \|u_t\|_{L^2(U_T)} \right)$$

$$\le C \cdot \left(\|f\|_{L^2(U_T)} + \|\psi\|_{H^1(\Sigma_0)} \right).$$

gain in regularity.